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Testing in Models of Asymmetric Information

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This paper explores the role of testing in models of asymmetric information. We demonstrate conditions under which testing for underlying characteristics can overcome adverse selection problems and lead to a full-information competitive equilibrium. This paper provides a more general statement of Mirrlees result on the optimal use of infinite fines. Where testing cannot fully resolve the problems associated with asymmetric information, we outline the source of the difficulties. Our results, developed in the context of a labour market, can be directly extended to other environments. In problems with asymmetric information, testing to discover an agent's chosen action or underlying characteristics may significantly reduce the cost of moral hazard and adverse selection.

1. INTRODUCTION

Models of self-selection illustrate the effects of asymmetric information in a wide range of economic environments. Examples of such environments are labour, capital, and insurance markets, taxation, regulation, education and law enforcement[1]. Typically, the absence of complete information about an agent's characteristics or chosen actions leads to a second-best equilibrium. An inefficiency arises because agents are indirectly motivated to behave honestly. Few studies (with the notable exception of Guasch and Weiss (1980), (1981), (1982a, b)) consider tests that directly discover an individual's true characteristics or actions. Such testing can reduce the cost of adverse selection and moral hazard. Indeed, we demonstrate conditions under which testing leads to the full-information competitive equilibrium.

The power of testing lies in its ability to provide information about the underlying characteristics of individuals. It helps determine whether their actions or signals represent truthful behaviour. Naturally, the effectiveness of testing as a self-selection device depends on its accuracy, its cost, the frequency of use, and the rewards for test performance. As always, economic tradeoffs must be considered. Greater accuracy may be possible only at higher cost. Lower frequencies of testing must be offset by larger penalties for failing, imposing costs on risk averse agents.[2]

These tradeoffs may lead to outcomes that deviate from the full-information allocation. It is not always possible to achieve a first-best allocation in models with asymmetric information and risk-averse agents; however, when the testing frequency, accuracy, and penalty size are coordinated optimally, under certain conditions, the full-information competitive equilibrium can be approached in the limit. This is accomplished by using a more expensive and more accurate test less frequently with an increased penalty for failure.

Three conditions must be satisfied to achieve the first-best: (i) risk-averse agents must know their own characteristics with certainty; (ii) agents' utility functions must be unbounded below; (iii) those characteristics initially unobservable to the principal must be verifiable ex post with arbitrarily high "accuracy" (in a sense to be made precise later) even if only at arbitrarily high cost.

These three conditions are satisfied in Mirrlees' (1974) model of moral hazard in the labour market. Agents know how hard they work and thus condition (i) is satisfied. Agents' utility functions are unbounded below, satisfying condition (ii). In Mirrlees' model an agent's effort is unobservable. The principal attempts to determine an agent's chosen effort by observing output. Mirrlees demonstrates conditions under which a very low output quota provides an arbitrarily accurate test of whether an agent chose the appropriate level of effort. Condition (iii) is therefore satisfied and the full-information outcome is approached in the limit.

In Mirrlees' model, output is costlessly observed. Because testing is free, all agents are tested. This paper generalizes Mirrlees' result to the case where testing is costly and used selectively. We consider a more general specification of the testing technology and show how Mirrlees' conditions for the optimality of a quota scheme are a special case of the conditions we place on the testing technology.

Our results are presented in the context of adverse selection in the labour market. Applications to other environments including those with moral hazard are straightforward. In our model, firms use a combination of tests and wage offers as a self-selection device. Both testing accuracy and the percentage of applicants tested are decision variables for the firm. In the limit, by testing workers with an arbitrarily small probability, very accurate tests and the threat of large penalties, the full-information competitive equilibrium is approached: workers receive wages equal to their marginal products and the existence of competitive equilibrium is guaranteed.

This result does not depend on workers being risk-neutral. Nor do we require that the tests are perfectly accurate. However, if testing accuracy is bounded, we show that

(a) the first-best allocation cannot be approximated;
(b) if an equilibrium exists, a strictly positive fraction of workers are tested and expected testing costs are positive.

Result (b) requires the additional assumption that firms are able to impose unbounded fines on workers.

In general, the fraction of applicants tested depends on the degree of risk aversion and the ratio of the maximum feasible fine to testing costs. Polinsky and Shavell (1979) show that if testing costs are sufficiently small then everyone will be tested. Guasch and Weiss (1982b) prove that even if it is feasible to impose unbounded utility fines, the optimal testing frequency may be strictly positive. We extend this result to demonstrate that if the ratio of the maximum feasible monetary fine to testing costs is sufficiently large, then workers will be tested with positive probability in equilibrium. A complementary result is proved by Bolton (1986): if the ratio of the maximum feasible fine to testing costs is sufficiently small, then workers are tested with arbitrarily small probability.

2. THE MODEL

Consider an environment with n types of agents, indexed by i and varying according to a parameter A_i. One can think of A_i as an individual's risk class in an insurance model, preference for a product in a discriminatory monopoly pricing model, or ability in an optimal income-tax model. We use the paradigm of risk-neutral firms operating in a competitive labour market in which A_i represents the worker's productivity. The special feature of our model is that a firm's information is not fixed and exogeneous; rather, a firm has available a testing technology which it can use to learn more about a worker's productivity. Although the setting we analyze is just one of many settings in which testing can be important—it is not crucial that we analyze a labour market or that it be competitive—we believe our chosen framework is easiest to follow and relates well to the existing literature.

We assume that a worker knows his productivity with certainty. His marginal product is the same both within and across firms. Firms know the distribution of productivity in the labour force; they do not know the productivity of any particular worker. For simplicity, we focus on the case where there are only two types of workers, labelled 1 and 2. Fraction α of the workers have high productivity, A_1, and the remaining fraction, $1-\alpha$, have low productivity, A_2. Let $\bar{A} \equiv \alpha A_1 + (1-\alpha)A_2$ be the average productivity of workers in this economy.

The structure of the market is as follows; in stage 1, each firm offers a single contract which a potential employee can either accept or reject.[3] In stage 2, each worker chooses at most one contract. Both the firm and the worker are bound to follow the terms of that contract. There is free entry of firms into the industry so that in equilibrium it is not possible for a firm to enter and earn positive profits. Since there is a constant returns to scale technology the equilibrium number of firms is indeterminate.

A contract in our model is not simply a wage that the firm pays the worker. Rather, a contract specifies

(i) a probability, θ, that the worker will be tested to determine (possibly inaccurately) his productivity;
(ii) an expenditure, t, on the test if it is administered;
(iii) wages conditional on the worker's test performance.

We assume that all of these variables are publicly observable and verifiable by the courts so that the contract can be enforced by either party in court.[4] It is assumed throughout that the firm and worker can commit not to renegotiate the contract after stage 2. The ability to commit is important. As we will show, the choice of a contract may reveal the worker's productivity so that after the contract is chosen it will not be in the interest of the firm to expend resources on testing the worker.

Each firm has access to the same testing technology. Let t denote the expenditure on testing a particular worker. The accuracy of the test is an increasing function of t; if a firm spends enough additional resources on testing a worker, it will be less likely to mistake a high-productivity worker for a low-productivity worker and vice versa. We assume that there are two possible outcomes of the test which we label "pass" and "fail". Let $p(t)$ be the probability that a high-productivity worker passes the test and let $1-q(t)$ be the probability that a low-productivity worker passes the test. The test is called uninformative if $p(t) = 1 - q(t) = 1/2$. In this case both types of workers are equally likely to pass or fail and test performance provides no information. We make the weak assumption, which we maintain throughout the paper, that for some positive, finite t, say $\hat{t}$, the test is informative; more productive workers are more likely to pass the test, $p(\hat{t}) > 1 - q(\hat{t})$.[5] Later, in Condition 1, we make a further assumption about the limiting properties of the testing technology.

Since test performance is publicly observable and verifiable, wages can be conditioned on test performance. Let W_p be the wage the firm pays the worker if he passes the test and let W_f be the wage if he fails. In the event that the worker is not tested, he is paid the passing wage, W_p, by assumption. This assumption simplifies the presentation without affecting the basic results. Note that were test performance not publicly observable and verifiable, firms would always claim that the worker failed if $W_f < W_p$ and claim he passed if $W_f > W_p$.

Thus, a contract, C, is a quadruple

$$C \equiv (\theta, t, W_p, W_f)$$

specifying a probability of being tested, $\theta \in [0, 1]$, an expenditure on the test, $t \in [0, \infty)$, and wages $W_p \in R$ and $W_f \in R$ conditional on test performance and whether the worker is tested. Define $S \equiv [0, 1] \times [0, \infty) \times R^2$ as the space of feasible contracts.

Consider now the conditions for equilibrium. Suppose M contracts are offered. Denote this set by $\Gamma_M \equiv \{C_1, C_2, \ldots, C_M\}$. Workers accept from Γ_M the contract that maximizes their expected utility. We assume all workers have the same risk averse von Neumann–Morgenstern utility function, $U(W)$, where $U'(W) > 0$ and $U''(W) < 0 \; \forall W$. Given utility maximization by workers, $\pi(C | \Gamma_M)$ denotes the profits of a firm offering contract C given that Γ_M is the set of contracts offered. A *free-entry Nash equilibrium* is defined as a set of M contracts $\Gamma_M^* \equiv \{C_1^*, C_2^*, \ldots, C_M^*\}$ such that

$$\text{for all firms offering contract } C_m^* \text{ and for all } m = 1, \ldots, M, \qquad (1)$$
$$\pi(C_m^* | \Gamma_M^*) \geqq \pi(C | \Gamma_m') \qquad \forall C \in S, \Gamma_m' = \{\Gamma_m^*, C\}^6,$$

$$\pi(C | \Gamma_M') \leqq 0 \qquad \forall C \notin \Gamma_M^*. \qquad (2)$$

The first condition is the Nash equilibrium condition which states that given the contracts offered, no firm can introduce a contract that increases its profits. The second condition follows from the assumption of free-entry; in equilibrium it must not be possible for a firm to enter and earn positive profits.

There are two possible equilibrium configurations. A *separating equilibrium* is defined as a free-entry Nash equilibrium in which low- and high-productivity workers accept different types of contracts. Let C_1^* be the contract accepted by high-productivity workers and let C_2^* be the contract accepted by low-productivity workers. In a separating equilibrium $C_1^* \neq C_2^*$. A *pooling equilibrium* is a free-entry Nash equilibrium in which both types of workers choose the same contract: $C_1^* = C_2^*$. Note that at most two contracts are offered in equilibrium[7].

If a high-productivity worker accepts a contract $C = (\theta, t, W_p, W_f)$ his expected utility, which we write as $V_1(C)$ is

$$V_1(C) \equiv [1 - \theta(1 - p(t))] U(W_p) + \theta(1 - p(t)) U(W_f).$$

With probability $(1 - \theta(1 - p(t))$, the worker receives wage W_p either because with probability $1 - \theta$ he is not tested or because with probability $\theta p(t)$ he is tested and passes. Alternatively, with probability $\theta(1 - p(t))$, the high-productivity worker who accepts this contract is tested, fails, and receives wage W_f.

If a low-productivity worker accepts contract, C, his expected utility, denoted by $V_2(C)$, is

$$V_2(C) \equiv (1 - \theta q(t)) U(W_p) + \theta q(t) U(W_f).$$

Here $q(t)$ is the probability that the worker fails the test if it is administered. Note that if $p(t) + q(t) > 1$ and $W_p > W_f$, then $V_1(C) > V_2(C)$ for all $C \in S$.

3. EQUILIBRIUM WITH TESTING

If tests were both perfectly accurate and costless, firms would always test to discover a worker's productivity. In reality, tests are both inaccurate and costly. One way of reducing expected testing costs is to test only a fraction of the workers who agree to work for the firm. But, if testing is infrequent, the penalty for failing must be large to discourage workers from misrepresenting their productivities. Given some inaccuracy in testing, these penalties expose workers to risk and hence reduce their expected utility if they are risk averse. Thus, there is a tradeoff between reducing testing costs by testing only a fraction of workers and exposing risk-averse workers to income variability. Although in general this tradeoff cannot be costlessly resolved, Proposition 2 demonstrates conditions under which the equilibrium outcome is arbitrarily close to the full-information competitive allocation. If the conditions for Proposition 2 are not met, an equilibrium may not even exist; if it does exist, the equilibrium differs from the full-information allocation. We begin by characterizing the properties that any equilibrium must satisfy and then discuss conditions under which equilibrium will exist.

The following proposition, which is similar to a well-known result in Rothschild and Stiglitz (1976), is useful in characterizing equilibrium.

Proposition 1. *No pooling equilibrium exists.*

Remark. The proof shows that at any putative pooling equilibrium, it would be possible for a firm to enter, offer a contract that attracts only high-productivity workers, and earn strictly positive profits, thus violating the definition of equilibrium.[7]

Proof. See Appendix. ||

Proposition 1 demonstrates that if an equilibrium exists it must separate different types of workers. The following lemmas describe necessary characteristics of any separating equilibrium. Proposition 2 then presents sufficient conditions for a separating equilibrium, and hence, any equilibrium to exist.

Lemma 1. *In any separating equilibrium, a contract is offered that pays a wage of* A_2, *involves no testing, and is accepted only by low-productivity workers*: $C_2^* = (0, 0, A_2, 0)$.

Proof. Suppose there is a separating equilibrium in which low productivity workers are tested. Since expected testing costs are strictly positive, the expected wage, $\bar{W}$, must be strictly less than A_2. But then a firm could enter and offer a contract with no testing and a guaranteed wage of $A_2 - \varepsilon > \bar{W}$. This contract earns positive profits as it attracts all low-productivity workers and might even attract high-productivity workers. Hence, the original contract could not have been part of a separating equilibrium. ||

Lemma 1 characterizes the contract that low-productivity workers will accept in a separating equilibrium. The equilibrium contract that attracts high-productivity workers maximizes their expected utility subject to two constraints: (i) only a high-productivity worker chooses that contract and (ii) profits from that contract are non-negative. Let $C_1^* = (\theta^*, t^*, W_p^*, W_f^*)$ be the contract that satisfies these conditions.[8] C_1^* solves the following program (P):

$$(P) \qquad \text{Maximize}_{\theta, t, W_p, W_f}\ U(W_p) - \theta(1-p(t))\Delta U$$

subject to

$$U(A_2) \geqq U(W_p) - \theta q(t)\Delta U \tag{3}$$

$$A_1 \geqq W_p - \theta(1-p(t))\Delta W + \theta t, \tag{4}$$

where

$$\Delta W \equiv W_p - W_f,$$

$$\Delta U \equiv U(W_p) - U(W_f).$$

Constraint (3) ensures that low-productivity workers prefer the certain wage of A_2 to the contract C_1^*. We refer to this as the self-selection constraint. Constraint (4) ensures that profits are non-negative.

The maximization problem is written in a way that separates utility into two parts: utility from the wage W_p minus the expected utility cost from testing, $\theta(1-p(t))\Delta U$ for high-productivity workers and $\theta q(t)\Delta U$ for low-productivity workers. If the test is uninformative $(p(t) = 1 - q(t) = 1/2)$, then these expected penalties are equal and it is impossible for a firm to offer a contract that attracts one type of worker but not the other. If, as assumed, the test is informative $(p(t) > 1 - q(t))$, then any contract involving testing is more attractive to high-productivity workers than low-productivity workers. As testing accuracy increases, $p(t)$ and $q(t)$ rise, the ratio of expected penalties increases and it becomes easier to offer a contract that high-productivity workers will accept but that will not attract low-productivity workers.

A necessary condition for existence of a separating equilibrium is that the program (P) has a solution. *A priori*, there is no reason to believe that both constraints (3) and (4) can be satisfied simultaneously. Lemma 2 demonstrates that if the test is informative $(p(t) + q(t) > 1)$ and testing costs are less than the difference in worker productivities $(\theta t < A_1 - A_2)$, then a solution to the maximization problem exists.

Lemma 2. *For a given (θ, t) such that $p(t) + q(t) > 1$ and $A_1 - \theta t > A_2$, the pair (W_p, W_f) which solves program (P) is the unique pair which simultaneously solves equations* (3) *and* (4).

Remark. The proof proceeds in two parts. We first show that both constraints must be binding at an optimum. Then we demonstrate that, under the conditions of the lemma, there is a unique solution to the pair of binding constraints.

Proof. See Appendix. ||

Lemmas 1 and 2 establish properties that any separating equilibrium must satisfy *if it exists*; however, an equilibrium may not exist. The following condition on the testing technology guarantees that a separating equilibrium exists and that it approaches the full-information competitive allocation.

Condition 1. $\lim_{t\to\infty} [1 - p(t)]/q(t) = 0.$

We define the "accuracy" of the test by the ratio in Condition 1. It is natural to assume that no finite expenditure will result in a perfectly accurate test $(p(t) = 1, q(t) > 0)$. Moreover, it seems likely that there would be decreasing returns to expenditures on testing: improving accuracy may require increasingly lage expenditures. Condition 1, however, only requires that type I errors are eliminated asymptotically. Of course, type I errors can be eliminated simply by passing everyone; thus, we require that $p(t)$ approaches 1 faster than $q(t)$ approaches zero.

Proposition 2. *Under Condition* 1, *a free-entry Nash Equilibrium exists, the equilibrium is separating, and the outcome approximates the full-information competitive equilibrium. Two contracts are offered: one offers a single wage of* A_2 *with no test; the other offers to test workers with arbitrarily small probability* $(\theta^* \to 0)$ *and arbitrarily large testing expenditure* $(t^* \to \infty)$, *and pays a wage* W_p^* *which approaches* A_1 *and a wage* W_f^* *which approaches* $-\infty$.

Remark. The basic structure of the proof is as follows. Let the expenditure per worker tested, t, approach infinity so that $p(t)$ approaches 1. Then choose the frequency of testing, θ, so that expected testing costs, θt, approach zero; θ must approach zero faster than t approaches infinity.[9] When θt is sufficiently small the conditions of Lemma 2 are satisfied and there exists a unique pair (W_p, W_f) satisfying both constraints. Observe that the ratio of expected utility fines for the high-productivity worker to the low-productivity worker is $(1-p(t))/q(t)$, which approaches zero by Condition 1. Even though $U(W_f)$ approaches negative infinity, the *expected* utility fine on a low productivity worker is bounded by $U(W_p)-U(A_2)$. Hence, the expected utility fine on the high-productivity worker is less than $[U(W_p)-U(A_2)](1-p(t))/q(t)$, which becomes arbitrarily small. His expected utility approaches $U(A_1)$, the same utility as under the full-information competitive equilibrium.

Proof. See Appendix. ||

Proposition 2 establishes that Condition 1 is sufficient for equilibrium to approach the full-information competitive equilibrium. Condition 1 is also *necessary* for this result to hold with strictly risk-averse workers. High ability workers bear no risk because $\theta(1-p)\Delta U$ approaches zero in the limit. If Condition 1 does not hold, then $\theta(1-p)\Delta U \to 0$ if and only if $\theta q \Delta U \to 0$. But if $\theta q \Delta U$ approaches zero the proposed equilibrium set of contracts will fail to separate workers as the self-selection constraint will be violated.

It is useful to compare the result in Proposition 2 to Mirrlees' (1974) result discussed earlier. In Mirrlees' model, agents choose an effort level, μ, which results in stochastic output $y=\mu+\varepsilon$ where ε is a random variable with cumulative distribution $F(\cdot)$. The agent's compensation is only a function of output since μ and ε are not observable.

Mirrlees considers the following compensation scheme. Let μ^* be the first-best level of effort. The agent is penalized if his output falls below some quota, X. In our model, we would say that he passes the test if his output is above X. Suppose the agent sets $\mu=\mu^*$. The probability of passing the test is $1-F(X-\mu^*)$. In our notation this is $p(0)$[10]. If the agent shirks by some amount, s, the probability of failing the test is $F(X-\mu^*+s)$ or in our framework $q(0)$.

By Proposition 2 if there is a test such that

$$(1-p(0))/q(0)=F(X-\mu^*)/F(X-\mu^*+s)=0, \tag{5}$$

then the full-information equilibrium is attainable. Now as $X\to-\infty$ the ratio in (5) approaches zero provided the hazard rate $f(X)/F(X)$ approaches infinity. This is precisely the condition Mirrlees imposes on the probability distribution to prove his result.

Proposition 2 generalizes Mirrlees' result to the case where observing output is costly and testing is random. We can reinterpret Mirrlees' condition on the hazard rate as

$$\Pr\{\text{mistaken penalty}\}/\Pr\{\text{correct penalty}\} \to 0.$$

which is Condition 1.

The relevance of this result does not depend on the possibility of actually using extreme punishments. The fact that the optimal testing scheme is only approached in the limit implies that when fines and utilities are bounded, the optimum may be at the boundary and not in the interior. This is more likely to be the case if the maximum fine is large.

When Condition 1 is not satisfied, Proposition 3 establishes that the equilibrium deviates from the first-best and θ does not approach zero in any separating equilibrium. Note, however, that Proposition 3 does not provide an existence result; under the conditions of the proposition an equilibrium may fail to exist. Thus, Proposition 3 simply characterizes the properties that any separating equilibrium must exhibit.

Proposition 3. *As income approaches negative infinity, let relative risk aversion be bounded away from zero, i.e.*

$$\lim_{W\to-\infty} WU''(W)/U'(W) = \bar{R} > 0.$$

Then for any fixed $\bar{t}$ for which $p(\bar{t}) < 1$, the probability of testing is strictly positive in any separating equilibrium.

Proof. See Appendix. ||

This result depends on both unbounded utility and unbounded fines. In contrast, Proposition 2 requires only unbounded utility fines[11]. If, for example, $U(W) = \ln(W + k)$, then the conditions for Proposition 2 are met while those for Proposition 3 are not; the maximum fine is bounded below because the utility function asymptotes at an income of $-k$.

Guasch and Weiss (1982*b*) demonstrate that if with unbounded utility fines are feasible, the equilibrium probability of testing need not approach zero. Proposition 3 demonstrates that the frequency of testing is strictly positive if the maximum fine is sufficiently large relative to testing costs. Bolton (1986) proves a complementary result: if the maximum fine is sufficiently small relative to testing costs, the probability of testing approaches zero.

One of the consequences of Proposition 3 is that an equilibrium may fail to exist. Given that there are positive expected testing costs and that high-productivity workers are exposed to risk, there may exist a pooling contract that makes both types of workers better off; but the pooling contract cannot be an equilibrium by Proposition 1. Hence, it may be possible to find a separating set of contracts that destroys any pooling equilibrium and a pooling contract that destroys any separating equilibrium (see Rothschild and Stiglitz (1976)).

A final point involves the way we model workers' information. Underlying our results is the assumption that workers know their own productivity with certainty. Suppose instead that a worker receives only a noisy signal of his productivity but that a high-productivity worker is more likely to receive a favourable signal. Ex post, only a worker's productivity can be verified by a test, not his prior belief about his productivity. Thus, even with accurate testing, workers will be exposed to some risk. However, the expected penalty on a worker receiving a favourable signal is finite and proportional to the probability that he overestimates his productivity. Thus, if Condition 1 holds and workers are nearly certain about their true productivity, then the equilibrium will be near the first-best allocation.[12]

4. CONCLUDING REMARKS

The purpose of this paper has been to explore the role of testing in models of asymmetric information. We demonstrated conditions under which testing for underlying characteristics can overcome adverse selection problems and lead to the full-information competitive equilibrium. This paper also generalises of Mirrlees (1974) result on the optimal use of infinite fines. Where testing cannot fully resolve the problems associated with asymmetric information, we outline the source of the difficulties. Our results, developed in the context of a labour market, can be directly extended to other environments. In problems with asymmetric information, testing to discover an agent's chosen action or underlying characteristic may significantly reduce the cost of moral hazard and adverse selection.

APPENDIX

Proposition 1. *No pooling equilibrium exists.*

Proof. There are two cases to consider: (a) a pooling equilibrium in which there is no testing and (b) a pooling equilibrium in which there is testing.

(a) Suppose there is a pooling equilibrium in which there is no testing. Then the wage must be $\bar{A}$. We show that there exists a contract in which workers are tested with positive probability that attracts only high-productivity workers and earns positive profits. The proof is divided into three steps:

Step 1. Suppose that $p(0)+q(0)>1$; at $t=0$, the test is informative. Let a firm offer this test with positive probability and make the following wage offer conditional on test performance:

$$W_p=\bar{A}+\delta, \qquad W_f=\bar{A}-\beta\delta.$$

The expected value of this wage offer to a low productivity worker is

$$\bar{A}+\delta[1-(1+\beta)\theta q].$$

Choose $1+\beta=1/\theta q$ so that the expected value of this wage offer is just $\bar{A}$. Since this is a lottery with mean $\bar{A}$ and the worker is risk averse, he will not accept the proposed contract.

On the other hand, *for small enough* δ, the high-productivity worker will accept this contract. To see this, take a Taylor series expansion around $\bar{A}$. His expected utility from accepting the contract is approximated arbitrarily closely by

$$U(\bar{A})+\delta U'(\bar{A})[1-\theta(1-p)(1+\beta)]=U(\bar{A})+(\delta/q)U'(\bar{A})[q+p-1]>U(\bar{A}).$$

His expected salary is greater than $\bar{A}$ since the test is informative. Hence, for arbitrarily small δ the high-productivity worker will accept this contract, whereas the low-productivity worker will not. Since the firm receives A_1 at an arbitrarily small cost above $\bar{A}$, the firm earns strictly positive profits, contradicting the assumption that the pooling contract was an equilibrium.

Step 2. Step 1 established that if testing is free, a contract can be offered that attracts only high-productivty workers and earns positive profits. But now suppose testing is not free but that there exists a (θ, t) pair such that $A_1-\theta t>\bar{A}$. Then the above contract still earns positive profits because, although the firm earns only $A_1-\theta t$ per worker, it pays out only marginally more than $\bar{A}$ in expected wages.

Step 3. By assumption there exists some finite $\bar{t}$ such that $p(\bar{t})+q(\bar{t})>1$. Choose θ such that $A_1-\theta\bar{t}>\bar{A}$. By Step 2, the proposed contract will earn positive profits. Hence, there can exist no pooling equilibrium without testing.

(b) Consider a pooling equilibrium in which there is testing. Suppose the proposed pooling contract is given by $\hat{C}\equiv(\hat{\theta}, \hat{t}, \hat{W}_p, \hat{W}_f)$. Consider the following offer $C'\equiv(\theta', t', W'_p, W'_f)$ where

$$\theta'=\hat{\theta}, \qquad t'=\hat{t}, \qquad W'_p=\hat{W}_p+\delta, \qquad W'_f=\hat{W}_f-\beta\delta.$$

For small δ, the expected utility to the low-productivity worker from accepting contract C' is approximated arbitrarily closely by

$$(1-\theta q)U(\hat{W}_p)+\theta qU(\hat{W}_f)+\delta[(1-\theta q)U'(\hat{W}_p)-\beta\theta qU'(\hat{W}_f)].$$

Choose

$$\beta=[(1-\theta q)U'(\hat{W}_p)]/[\theta qU'(\hat{W}_f)]$$

so that the low-productivity worker is indifferent between contract C' and contract $\hat{C}$. In this case, the high-productivity worker prefers contract C' over contract $\hat{C}$. His expected utility from C' is approximated by

$$[1-\theta(1-p)]U(\hat{W}_p)+\theta(1-p)U(\hat{W}_f)+\delta\{[1-\theta(1-\theta(1-p)]U'(\hat{W}_p)-\beta\theta(1-p)U'(\hat{W}_f)\}$$

If the last term is positive then high-productivity workers prefer contract C'. Substituting in the above value of β shows the final term is positive as

$$U'(\hat{W}_p)\delta[q+p-1]/q>0$$

since the test is informative. Hence, there exists a contract that attracts only high-productivity workers, paying them expected wages slightly in excess of $\bar{A}-\theta t$ and earning $A_1-\theta t$ per worker. Since $A_1>\bar{A}$, this contract earns positive profits contradicting the assumption that the original contract was part of an equilibrium. ||

Lemma 2. *For a given (θ, t) such that $p(t)+q(t)>1$ and $A_1-\theta t>A_2$, the pair (W_p, W_f) which solves program (P) is the unique pair which simultaneously solves equations* (3) *and* (4).

Proof.
Step 1. Consider the Lagrangian to the optimization problem (P),

$$L=U(W_p)-\theta(1-p(t))\Delta U+\gamma[U(A_z)-U(W_p)+\theta q(t)\Delta U]+\mu[A_1-W_p+\theta(1-p(t)\Delta W-\theta t],$$

where γ and μ are the Lagrange multipliers on constraints (3) and (4), respectively. Inspection of the first-order conditions reveals that both γ and μ must be positive.

$$\partial L/\partial W_p=[1-\theta(1-p)]U'(W_p)-\gamma(1-\theta q)U'(W_p)-\mu[1-\theta(1-p)]=0 \tag{A.1}$$

$$\partial L/\partial W_f=\theta(1-p)U'(W_f)-\gamma\theta qU'(W_f)-\mu\theta(1-p)=0 \tag{A.2}$$

where for shorthand we write p and q instead of $p(t)$ and $q(t)$.

(a) If both constraints were slack, then it would be optimal to set $W_f=\infty$ and $W_p=\infty$. But this violates the profit constraint (4).

(b) Suppose only constraint (4) is binding. Then $\gamma=0$ and (A.1) and (A.2) together imply $U'(W_p)=U'(W_f)=\mu$ so that $W_p=W_f$. But if $W_p=W_f$ and (4) is binding, then $W_p=W_f=A_1-\theta t$. By assumption $A_1-\theta t>A_2$; therefore, the self-selection constraint (3) is violated.

(c) Suppose only constraint (3) is binding so that $\mu=0$. Then the first-order conditions (A.1) and (A.2) reduce to:

$$\partial L/\partial W_p=\{[1-\theta(1-p)]-\gamma(1-\theta q)\}U'(W_p)=0 \tag{A.3}$$

$$\partial L/\partial W_f=\{(1-p)-\gamma q\}U'(W_f)=0 \tag{A.4}$$

Solving (A.4) for γ and substituting into (A.3) yields the contradiction

$$\partial L/\partial W_p=(1/q)[q+p-1]U'(W_p)>0 \tag{A.5}$$

since the test is informative.

Hence, both γ and μ are strictly positive which implies that both constraints must be binding at an optimum.

Step 2. We now show that for a given (θ, t) such that $A_1-\theta t>A_2$ there is a unique (W_p, W_f) pair that solves equations (3) and (4). Equation (4) defines a linear relationship between W_f and W_p:

$$W_f(W_p)=\frac{A_1-\theta t-[1-\theta(1-p)]W_p}{\theta(1-p)}.$$

Substituting this expression into (3), we can write the low-productivity worker's utility solely as a function of W_p. Denote this utility by $V_2(W_p)$. An increase in W_p lowers the low-productivity worker's utility for two reasons: (i) The resulting adjustment to W_f is actuarially fair to the high-productivity type, but unfair to the low-productivity type; (ii) The increased spread between W_p and W_f imposes greater risk:

$$\frac{dV_2}{dW_p}=(1-\theta q)U'(W_p)-q[1-\theta(1-p)]U'(W_f(W_p))/(1-p) \tag{A.6}$$

$$\leqq -U'(W_f(W_p))[p+q-1]/(1-p)<0. \tag{A.6'}$$

The inequality follows as $W_p \geqq W_f(W_p)$ implies $U'(W_p) \leqq U'(W_f(W_p))$ and $p+q>1$ implies $q[1-\theta(1-p)] > (1-p)(1-\theta q)$. Moreover, as the spread between W_p and W_f increases, $V_2(W_p)$ falls at an increasing rate. At $W_p = A_1 - \theta t$, the self-selection constraint is violated: $W_f = W_p$ and $V_2(W_p) > U(A_2)$ as $A_1 - \theta t > A_2$. By (A.6), since $V_2(W_p)$ monotonically falls (at an increasing rate), there must exist a unique solution to $V_2(W_p) = U(A_2)$. ||

Proposition 2. *Under Condition 1, a free-entry Nash Equilibrium exists, the equilibrium is separating, and the outcome approximates the full-information competitive equilibrium. Two contracts are offered: one offers a single wage of A_2 with no test; the other offers to test workers with arbitrarily small probability ($\theta^* \to 0$) and arbitrarily large testing expenditure ($t^* \to \infty$), and pays a wage W_p^* which approaches A_1 and a wage W_f^* which approaches $-\infty$.*

Proof. Choose $t = 1/\sqrt{\theta}$ and let $\theta \to 0$. As $\theta \to 0$, $t \to \infty$, but $\theta t \to 0$. Since $t \to \infty$, $[1-p(t)]/q(t) \to 0$.

Applying Condition 1 to equation (A.6′) in the proof of Lemma 2 shows that the absolute value of $dV_2(W_p)/dW_p \to \infty$ as $t \to \infty$:

$$|dV_2(W_p)/dW_p| \geqq \frac{U'(W_f(W_p))[p(t)+q(t)-1]}{1-p(t)},$$

$$\geqq \frac{U'(A_1)[p(t)+q(t)-1]}{1-p(t)}.$$

Therefore,

$$\lim_{t\to\infty} |dV_2(W_p)/dW_p| \geqq U'(A_1) \lim_{t\to\infty} |q(t)/(1-p(t))-1| \to \infty.$$

Recall from the proof of Lemma 2 that at $W_p = A_1 - \theta t$, $V_2(W_p) = U(A_1 - \theta t)$ and the self-selection constraint is violated. Since the absolute value of the derivative of $V_2(W_p)$ becomes arbitrarily large at all values of $W_p \geqq A_1 - \theta t$, and $dV_2(W_p)/dW_p < 0$, an infinitessimal increase in W_p lowers $V_2(W_p)$ down to $U(A_2)$, satisfying the self-selection constraint. In the limit, W_p must be arbitrarily close to $A_1 - \theta t$ and thus to A_1 as well since $\theta t \to 0$.

Now rewrite the high-productivity worker's expected utility, defined by V_1, using the self-selection constraint as an equality:

$$V_1 = U(W_p) - [(1-p(t))/q(t)]\{U(W_p) - U(A_2)\}$$

Taking the limit as $t \to \infty$ yields, $V_1 \to U(A_1)$ since by Condition 1, $(1-p)/q \to 0$ while $U(W_p) - U(A_2)$ is bounded.

Thus, we have shown that low productivity workers accept the contract that guarantees them a wage of A_2 and high-productivity workers accept a contract which guarantees them an expected utility that comes arbitrarily close to $U(A_1)$. Finally, we note that no pooling contract can break this equilibrium since it is not profitable to pay both types of workers A_1. ||

Proposition 3. *As income approaches negative infinity, let relative risk aversion is bounded away from zero, i.e.*

$$\lim_{W\to-\infty} WU''(W)/U'(W) = \bar{R} > 0.$$

Then for any fixed $\bar{t}$ for which $p(\bar{t}) < 1$, the probability of testing is strictly positive in any separating equilibrium.

Proof. We proceed by contradiction and show that the limiting contract as $\theta \to 0$ cannot be an equilibrium. Assume that the equilibrium occurs in the limit as $\theta \to 0$. Repeated application of l'Hopital's rule leads to the contradiction that for the Lagrangian in Lemma 2, $\partial L/\partial\theta$ is positive in the limit as $\theta \to 0$.

The necessary first-order condition for θ is given by

$$\partial L/\partial\theta = -[1-p(\bar{t}) - \gamma q(\bar{t})]\Delta U + \mu[(1-p(\bar{t}))\Delta W - \bar{t}] = 0 \tag{A.7}$$

Substituting the first-order condition for W_f, (A.2), into the above equation yields,

$$\partial L/\partial\theta = \mu(1-p(\bar{t}))\left\{\frac{\Delta W U'(W_f) - \Delta U}{U'(W_f)} - \frac{\bar{t}}{1-p(\bar{t})}\right\}. \tag{A.8}$$

As monitoring becomes less frequent, the penalty for failing must become increasingly severe to satisfy the self-selection constraint: $\theta \to 0$ implies $W_f \to -\infty$. To demonstrate that $\partial L/\partial\theta$ is positive near $\theta = 0$, we show that under the conditions of the proposition, the first term in brackets, which we denote by Z, becomes unbounded as $\theta \to 0$. The numerator of Z, $\Delta WU'(W_f) - \Delta U$, is infinite in the limit since the first term approaches infinity faster than the second term. To see this, note that by l'Hopital's role:

$$\lim_{W_f \to -\infty} \frac{\Delta WU'(W_f)}{\Delta U} = \lim_{W_f \to -\infty} \frac{\Delta WU''(W_f)U'(W_f)}{-U'(W_f)}$$

$$= 1 + \lim_{W_f \to -\infty} [(W_f - W_p)/W_f][W_f U''(W_f)/U'(W_f)]$$

$$= 1 + \bar{R}. \tag{A.9}$$

The denominator of Z also approaches infinity. It remains to be shown that the numerator approaches infinity faster than the denominator. Again, by l'Hopital's rule,

$$\lim_{W_f \to -\infty} \frac{\Delta WU'(W_f) - \Delta U}{U'(W_f)} = \frac{\Delta WU''(W_f)}{U''(W_f)} = \Delta W = \infty. \tag{A.10}$$

Substituting the result of (A.10) back into (A.8) reveals

$$\lim_{W_f \to -\infty} \partial L/\partial \theta = \mu(1 - p(\bar{t}))\left\{\Delta W - \frac{\bar{t}}{1 - p(\bar{t})}\right\} > 0. \tag{A.11}$$

Hence for some θ^*, $\partial L/\partial \theta > 0$ for all $\theta < \theta^*$. This contradicts the assumption that $\theta = 0$ is the limiting value of the solution to the program (P), when t is constrained to equal $\bar{t}$. ||

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NOTES

1. For examples of this literature see Mirrlees (1976), Stiglitz (1982) (optimal income taxation), Guasch and Weiss (1981), Nalebuff and Stiglitz (1983) (labour markets), Stiglitz and Weiss (1981) (capital markets), Rothschild and Stiglitz (1976) (insurance markets), and Stiglitz (1975), Spence (1974) (education).
2. This contrasts with the simplest specification of the law-enforcement problem. For crimes without social value, the punishment is used as a deterrent and should be large enough to prevent all crimes. Thus, in equilibrium fines are never imposed and hence they do not impose risk on anyone. See Becker and Stigler (1974) and Polinsky and Shavell (1979).
3. We focus on the case where each firm can make only one offer; allowing firms to make multiple, simultaneous offers would not affect any of the results.
4. A variable is said to be *publicly observable* if all parties to the contract can observe it; a variable is said to be *verificable* if it is observable by the courts. Here, the optimal contract would specify values for these variables and a penalty if the contract is breached. Hence, provided court costs are relatively small, there will never be breach in equilibrium because all breaches will be litigated.
5. This testing technology differs from the one analyzed in Gausch and Weiss (1982a). In their model, tests are perfectly accurate in the sense that all workers who pass the test have ability above some cutoff point. Firms choose the passing grade, not the precision of the test. Guasch and Weiss analyze the distribution of cutoff points across firms.
6. In examining competitive equilibrium, we assume that each contract is offered by more than one firm. As a result, if one firm offers contract C rather than C_m^*, workers do not lose the option of C_m^*. Thus, Γ_m has all of the contracts in Γ_m^* and C.
7. Here we ignore (semi-pooling) equilibria in which, with positive probability, different types of workers accept the same contract. It is straightforward to generalize Proposition 1 to show that no semi-pooling equilibria exist.
8. These conditions are only necessary and not sufficient. Given C^*, no other separating contract earns positive profits. However, profitable pooling contracts may exist.
9. Note that the actual limit of the sequence $\theta \to 0$ is not an equilibrium since there is no testing.
10. It is $p(0)$ since output is costlessly observable and the test is free.
11. The proof of Proposition 3 depends on taking limiting values as $W_f \to -\infty$. This approach is invalid if ΔW is bounded.
12. A more formal discussion of this issue can be found in our working paper, Nalebuff and Scharfstein (1984).

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