TO THE RAIDER GOES THE SURPLUS? A REEXAMINATION OF THE FREE-RIDER PROBLEM

BENGT HOLMSTRÖM

AND

BARRY NALEBUFF

School of Organization and Management
Yale University
Box 1A, Yale Station
New Haven, CT 06520

This paper reexamines Grossman and Hart's (1980) insight into how the free-rider problem excludes an external raider from capturing the increase in value it brings to a firm. The inability of the raider to capture any of the surplus depends critically on the assumption of equal and indivisible shareholdings—the one-share-per-shareholder model. In contrast, we show that once shareholdings are large and potentially unequal, a raider may capture a significant part of the increase in value. Specifically, the free-rider problem does not prevent the takeover process when shareholdings are divisible.

I. INTRODUCTION

Takeovers act as an important disciplinary device in the market place. When a company's stock price falls below its potential value, there is an incentive to take over the company, change its operations, and increase the stock's valuation. The $64-million question is, Who gets the surplus?

Here we are not asking who *should* get the surplus. We are asking under the current rules for takeover, how will the surplus value be distributed between current shareholders and the value-improving new management?

The insight of Grossman and Hart (1980) is that the free-rider problem may completely exclude an external raider from capturing any of the increase in value.[1] The profit motivation for takeovers has relied on raising the value of the raider's own holdings (Shleifer and Vishney, 1986), or an ability to use control in order to dilute the value of minority shareholder's interests (Bebchuk, 1988; Grossman and

1. We review their argument in the next section.

Journal of Economics & Management Strategy, Volume 1, Number 1, Spring 1992

Hart, 1980). These may all be thought of as *private* values because the raider's interest in gaining control reflects expected personal gains.

Private values are an important reason to engage in takeover activity and are often adequate to compensate a raider for search and transaction costs. But these are small pieces of a big pie. A raider's stake is typically between 5% and 15% of the company. While this provides an incentive to play the game, it suggests that too little takeover activity will occur: The raider is paying the full cost but only getting a small fraction of the benefit.

This article looks at how the raider can capture more of the pie. The focus is on the free-rider problem. *Our results show that the raider need not give away more than 50% of the surplus on shares the raider does not own.*[2] This is true even when dilution is impossible, and the raider has no stake in the company.

The difference between our model and that of Grossman–Hart is that we do not approximate a large number of shareholders by a continuum. Instead, we study the equilibrium with a finite number of shareholders and examine the raider's potential surplus as that number grows large. When each shareholder has only one share, the free-rider phenomenon is confirmed: the raider makes no surplus in the limit.[3] But, one share per shareholder is the worst possibility for the raider. Tendering is an all-or-nothing decision, so there is no possibility of splitting the surplus.

To demonstrate that free-riding is due to this indivisibility problem, we consider the effect of repeated "stock-splits": the number of shares held by each shareholder grows, while the total value of the company remains constant. The stock-split helps the raider because there is a greater incentive to tender when a person holds a larger number of shares. A share tendered raises the chance of success and, thus, the value of all remaining untendered shares. This externality is valued in proportion to the number of shares remaining. Thus, a person holding many shares has a higher marginal incentive to tender than someone with only one share. The person with one share cares only about the tender price, because the increased chance of success is of no value once the one share is tendered.

In the limit, as shareholdings become infinitely divisible, the raider and the shareholders split the surplus. Shareholders are asked

2. This assumes a 50% majority is needed for control. If the percentage needed for control is higher, say 64%, then the raider can keep 64% of the surplus and need give away only 36% to the shareholders.

3. This result first appears in Bagnoli and Lipman (1987).

to tender half their holdings (for free), and in return they keep the rise in value on their remaining half.

This striking difference from Grossman–Hart suggests that there are two distinct results one could arrive at depending on how one takes the limit. Taking the limit as the number of shareholders becomes large leads to zero raider surplus. Taking the limit as the shares becomes increasingly divisible leads to 50% of the surplus going to the raider. If one allows both the number of shareholders and the number of shares to grow simultaneously, the raider's surplus still approaches 50%, provided the number of shares per shareholder grows faster than the square root of the number of shareholders. For most widely held companies, this seems to be a reasonable approximation.[4]

Consideration of the finite number of shareholders model leads to other surprises. For example, a common anti-takeover tactic is for a company to adopt a super-majority (greater than 50% majority) rule for attaining control. By requiring the raider to capture more shares, this tactic is meant to make takeovers more difficult and more costly. We show just the opposite. A super-majority rule increases the expected surplus going to the raider. The mistaken intuition looks only at the raider's demand equation: because the raider's demand rises, the raider is thought to pay a higher price. But there is a supply effect too. Shareholders recognize that a super-majority increases the chance they will be pivotal. This provides a greater incentive to tender that more than compensates for the raider's increased demand.

It is important to emphasize that our paper is designed to explore the logical implications of a model rather than making empirical predictions. Thus, we do not include a measure of a raider's private valuation, the size of the raider's stake, nor the raider's ability to dilute the value of minority shareholders. These complications provide important additional incentives for engaging in takeover activity. Our main point is that even without these incentives, a raider may still capture part of the increase in value: *the free-rider problem does not prevent the takeover process when shareholdings are divisible.*

Before we proceed with the formal model, it is worthwhile to review some of the mechanical details surrounding tender offers. There are three basic types of offers. The simplest and increasingly common type of tender offer is a bid for any and all shares. This is called an *unconditional* offer. The raider agrees to buy however few or many shares are tendered. A second type of bid is called a *conditional*

4. With 250,000 shareholders each holding 1,000 shares, this ratio is 2. Even a ratio of 2 is a big number—it implies that the raider's probability of success is above 90%.

tender offer. The offer to buy shares is conditional on the raider receiving at least some minimum number being tendered, typically controlling interest. If fewer shares are tendered, the raider is under no obligation to purchase any. The third type of bid is a *restricted* offer. The restriction states an upper bound on the number of shares the raider is willing to accept. If more than the upper bound are tendered, then the raider must prorate purchases, buying an equal fraction from each of the tenderers so as to achieve the desired total.[5]

Given this variety of tender offers, is there one type that has a comparative advantage in circumventing the free-rider problem?[6] We show that there is a fundamental equivalence between unconditional, conditional, and restricted offers.[7] With an appropriate adjustment in the bid, the raider's expected surplus and the chance of a successful takeover do not depend on the type of offer used.

The plan of this paper is as follows: in Section 2, we briefly review the Grossman–Hart argument. Section 3 translates their findings into the large, but finite, number of shareholder economy. Section 4 presents our main results for economies where shareholders each have a large number of shares. Our conclusions are in section 5. The proofs for most of the results are in the Appendix.

2. The Grossman—Hart Model

Small shareholders have an ability to free ride on a raider's attempt to improve the value of a company. The argument follows Grossman and Hart (1980). For simplicity, let the current value of the company be 0. If the raider gains control, the value will rise to 1. To approximate the fact that most companies are widely held, there are a continuum of shareholders, each with one "share." Consider what will happen if the raider attempts a takeover with an unconditional tender offer of v, $0 < v < 1$.[8]

Can this offer succeed with probability one? If this is an equilibrium, shareholders can predict this success. Any shareholder should

5. These second and third options may be combined in a *restricted-conditional* offer.

6. See, for example, Bebchuk (1988), who argues that unconditional tender offers work better than conditional offers in capturing surplus for the raider.

7. This equivalence requires that the conditional offer is conditioned on the raider gaining control. See the discussion following Proposition 2.

8. The tender offer v reflects a price for the entire firm. If shareholders tender mass m, they are given payment mv. If all shares are tendered, the mass adds up to one, and the raider pays v.

then refuse to tender at v, because he can earn 1 by holding out.[9] Because shareholders are a continuum, no one individual believes the decision not to tender his share will affect the outcome. But like Zeno's grains of sand, everyone's refusal to tender adds up to failure for the raider.[10] To succeed, the raider must offer what shareholders believe the price will be following the successful takeover. But if the raider offers $v = 1$, the raider's successful takeover will be an empty victory.

Behind this argument lies a paradox. If an offer $0 < v < 1$ is to succeed, the free-rider problem dooms it to failure. What if shareholders expect it to fail? In that case holding out is worth 0, while tendering to the unconditional bid is worth $v > 0$. By the same reasoning as earlier, everyone now wants to tender with the result that the bid will succeed. This suggests that no pure strategy solution, neither certain success nor certain failure, can be a equilibrium for unconditional tender offers $0 < v < 1$. The very fact that everyone expects a result leads to the opposite conclusion.

To find an equilibrium, we most consider mixed strategies in which the outcome is left to chance. This is required because everyone wants to be negatively correlated with the majority. When the majority tenders, everyone wants to hold out, while when the majority holds out, everyone want to tender. This problem is not unique to tender offers. Vacation travelers seeking a quiet unspoiled resort face a similar problem. If everyone wants to go to the resort with the fewest people, this becomes a self-negating prophecy. As Yogi Berra put it, "The club is so popular, no one goes there anymore." The problem is that a majority can't hide from the crowd.

With a continuum of shareholders, it is hard to know how to proceed: the aggregate percentage tendered is not a random variable. The mixed-strategy equilibrium does not appear well defined. Our solution is to abandon the continuum. It is an approximation of reality chosen to simplify the mathematical arguments. Because it is now adding complications, we return to a model with a finite number of shareholders and consider the possibility of raider surplus as the number of shareholders becomes large.

9. The argument is even stronger when the bid is conditional. Not tendering is a weakly dominant strategy: Why accept $v < 1$ only in the circumstances when the raider succeeds and your stock becomes worth 1?

10. In Zeno's paradox, adding one grain of sand can never transform a nonheap into a heap. Yet with enough grains, even a molehill can be turned into a mountain.

3. The One-Share-per-Shareholder Model

3.1. An Unconditional Tender Offer

Consider a world where there are N risk-neutral shareholders, each of whom owns exactly one share. In order to take over the company, the raider must have control of K shares. Typically, 50% majority is needed for control so that $K = [(N + 1)/2]$. The raider makes an unconditional tender offer of $0 < v < 1$. Again, v is the offer for the entire firm so that each share tendered gets v/N.

The number of Nash equilibria is now quite large. First, there are $\binom{N}{K}$ pure strategy equilibria in which the raider always succeeds. In these equilibria, exactly K out of the N shareholders are "designated" to tender while the other $(N - K)$ are allowed to free ride. Everyone is choosing their best response. If any designated tenderer deviates, the offer fails leading to a payoff of 0 rather than $v > 0$. If a free rider deviates by tendering, the offer still succeeds, but this lowers his or her payment to $v < 1$.[11] Because there are $\binom{N}{K}$ ways of choosing the K people to tender, each one corresponds to a different equilibrium. Of course, "designating" K volunteers is an extreme solution to the free-rider problem. In the decentralized marketplace, there is no way to coordinate which K individuals are the ones who must tender and which $(N - K)$ get to free ride. In fact, it would be illegal to make the tender offer only to those holding even numbered stock certificates (or any other identification).

There are an even greater number, $\sum_{i=0}^{K-1} \sum_{j=K+1-i}^{N-i} \binom{N}{i}\binom{N-i}{j}$, of partial mixed strategy equilibria in which i shareholders tender with probability 1, $N - i - j$ tender with probability 0, and the remaining j pursue a mixed strategy. Here, too, one faces the question of how to choose which shareholders get to play each of these roles.

Among these many partial mixed strategy equilibria, there is one and only one in which all shareholders pursue a mixed strategy. This mixed strategy equilibrium is symmetric, and, thus, the expected payoffs are equal across shareholders. Uniqueness and symmetry makes this equilibrium a focal point. Unlike all the pure and semi-mixed strategy solutions, there is no need for coordination between the shareholders. This solution is the starting point for our model.

11. Note that these equilibria do not depend on v so that the raider can capture arbitrarily close to K/N of the surplus. See Bagnoli and Lipman (1988) for greater detail on this approach.

3.2 The Symmetric Equilibrium

If a shareholder is to pursue a mixed strategy, he must be indifferent between tendering and holding on to his share. Imagine that the probability of success equals the fraction of the surplus the raider is giving away, v. The value of free-riding is $1/N$ with probability v. This has to be compared with the unconditional payment of v to those who tender.

Tender	Don't Tender
Get $\frac{v}{N}$ for certain.	Get $\frac{1}{N}$ with probability v.

A risk-neutral shareholder is indifferent between these options. Hence, the shareholder is willing to pursue a mixed strategy and tender with probability p. If p is sufficiently close to one, the chance of success approaches 1; if p is sufficiently small, the chance of success goes to zero. Somewhere in between is a probability p such that if everyone tenders with chance p, the aggregate probability of success will be v.

What surplus does that leave for the raider? We could calculate the raider's surplus directly, but there is an easier way. The marketplace is a closed system. All the surplus goes either to the raider or the shareholders. The raider captures the total surplus net of what the shareholders get. Expected total surplus is easy to calculate: it is simply the chance of a successful takeover, v. Each shareholder expects surplus of $\frac{v}{N}$, which adds up to a total of v. It appears there is nothing left over for the raider.

$$\text{Raider Surplus} = v - v = 0. \tag{1}$$

The previous argument is not quite correct. The problem is that the shareholder's comparison left out his probability of being pivotal. Specifically, in the event there are $K - 1$ tenders out of the other $N - 1$ shareholders, the *Nth* shareholder wants to tender at any positive v. If he tenders, he is paid v unconditionally, while if he does not tender, the offer fails, and he gets zero. Although the probability of this event is small, it destroys the indifference argument in the previous section. It is easy to see this in an example with three shareholders.

Example 1. Let $N = 3$, $K = 2$, and $v = \frac{1}{2}$. A shareholder's value of free-riding equals the chance of success *conditional* on his not tendering. Each shareholder tenders with chance p in equilibrium. With one free-rider, the offer succeeds only if both other shareholders tender, which has chance p^2. To make this equal to the value of tendering, it follows that $p^2 = 0.5$. Hence, $p \approx 0.7$, which leads to an aggregate chance of success $3p^2(1 - p) + p^3 \approx 0.8$. By the adding-up constraint, there is an

80% chance of success, but the shareholders are only given $v = \frac{1}{2}$. That leaves 30% for the raider. The 30% reflects the probability an individual is pivotal and tenders.

Although 30% is a large amount of surplus for the raider, three is a very small number of shareholders. As the number of shareholders grows, the chance that any one individual is pivotal shrinks to zero. But that does not mean it can be ignored. If everyone discounts the chance they will be pivotal, then as suggested in the first model, there is no surplus left for the raider.[12]

The one-share-per-shareholder model is not meant to solve the free-rider problem. It is used only to get our foot in the door. Once we have found some positive surplus, the next step is to magnify the raider's share until it reaches 50%. Magnifying zero doesn't help, and that is why establishing the existence of positive surplus is an essential first step. We proceed with the one-share-per-shareholder case, confirming the Grossman–Hart result for large finite number of shareholders. Once we have developed the mathematical framework, we move to the more interesting but complicated case where shareholdings are not necessarily equal across shareholders.

Let $P(T \geq K)$ denote the probability the tender offer succeeds, that is, the chance that the number of shares tendered, T, is at least K out of N. A shareholder that attempts to free ride must realize that the chance of success is reduced to $P(T_{-i} \geq K)$, the probability of getting at least K shares from the remaining $N - 1$ shareholders. Thus, in response to an unconditional offer of v, shareholders are indifferent between tendering and free riding if and only if $P(T_{-i} \geq K) = v$.

Tender	Don't Tender
Get $\frac{v}{N}$ for certain.	Get $\frac{1}{N}$ with probability $P(T_{-i} \geq K)$

The expected total surplus is $P(T \geq K)$. The raider gives up v, which is equal to $P(T_{-i} \geq K)$.

$$\begin{aligned}\text{Raider Surplus} &= P(T \geq K) - P(T_{-i} \geq K) \\ &= pP(T_{-i} = K - 1) \\ &= \binom{N-1}{K-1} p^K (1-p)^{N-K}.\end{aligned} \tag{2}$$

12. This result differs from Bebchuk (1988). His calculations are based on his assumption that "shareholders ignore the possibility that their decision will determine the outcome of the bid." He goes on to claim that "dropping this assumption will make the analysis more complicated but would not change its conclusions." In fact, the possibility of being pivotal is critical to the finite shareholder model.

Here p is the chance any one shareholder tenders and $P(T_{-i} = K - 1)$ denotes the probability that a shareholder is pivotal, that is, exactly $K - 1$ shares are tendered by the other $N - 1$ shareholders. The surplus left for the raider equals the chance that a given shareholder is pivotal and then tenders so that the offer succeeds. As written, p is equal across i; the mixed strategy solution is unique and symmetric (as demonstrated by Lemma 1 of the Appendix).

Although it is standard to think of the raider choosing the tender offer v and then finding the equilibrium p, it is simpler to reverse this process. Imagine that the raider picks a p he wants to motivate. This determines the corresponding equilibrium v. Because there is a one-to-one mapping from p to v, the two ways of looking at the problem are equivalent. To maximize raider surplus, the raider chooses p^*, the solution to the first-order condition below.[13]

$$\frac{K}{p^*} - \frac{N-K}{(1-p^*)} = 0 \textit{ or } p^* = \frac{K}{N} \tag{3}$$

$$\text{Raider Surplus} = \binom{N-1}{K-1}\left(\frac{K}{N}\right)^{K}\left(\frac{N-K}{N}\right)^{N-K} \tag{4}$$

The raider chooses p^* so that the expected number of shares tendered is just the number needed. This maximizes the probability that any individual will be pivotal. At this point we are ready to answer three of our initial questions: What is the limiting surplus as the number of shareholders gets large? What is the effect of a supermajority rule? How are conditional and restricted offers different?

3.3 Free Riding with a Large Number of Shareholders

We can use a normal approximation to the binomial to estimate the raider's surplus. The sum of N independent shareholder decisions, each tendering with probability $p^* = K/N$ has expectation K and variance $Np^*(1 - p^*)$. The raider's surplus is p^* times the chance a shareholder is pivotal or approximately p^* times the chance of falling in the range $K - 1$ to K:

$$\text{Raider Surplus} \approx \sqrt{\frac{p^*}{2\pi N(1-p^*)}}\,. \tag{5}$$

13. One can see directly from eq. (3) that for $p < p^*$, the first-order condition is positive, while for $p > p^*$, the first-order condition is negative. Hence, the optimization problem is strictly quasi-concave, and the solution to the first-order condition must be the global maximum.

It is immediate that as $N \to \infty$, the raider surplus approaches zero. The surplus falls slowly, at rate $\frac{1}{\sqrt{N}}$.[14] If $N = 100$ and 50% majority is needed for control, then surplus is roughly 4%, while if $N = 10{,}000$, raider surplus is roughly 0.4%.

Bagnoli and Lipman (1988) present an equivalent calculation that confirms this Grossman and Hart free-rider problem. Raider surplus goes to zero when the number of shareholders increases, while the value of the firm remains constant. They observe that there is still a possibility for raider surplus when the value of the firm is correlated with the number of shareholders. For example, if all shares are worth 1, the entire firm is worth N, and the raider captures surplus equal to $\sqrt{N}$, which grows with N. To the extent that more widely held companies are also more valuable, surplus to the raider is possible even in a one-share-per-shareholder model. But note that the raider's *fraction* of the increase in value still goes to zero, and, thus, there is too little incentive to pursue takeovers.

3.4 The Poison-Pill That's Good for the Raider

The anti-takeover tactic of super-majority rule is to raise the control cutoff K. This is meant to lower the chance of success and reduce the surplus available to the raider. In fact, it has just the opposite effect in our model.

Proposition 1: *For $K < N$, raising the control majority increases the raider's surplus.*

In the limit when $K = N$, the raider surplus is 100%. Under 100% majority every shareholder is pivotal. Any person who fails to tender

14. So far we have focused on the mixed-strategy solution. There is a clear sense in which this is a knife-edge result. If some of the people were not tendering with exactly the right probability, then the resulting chance of being pivotal would be much, much smaller. Chamberlain and Rothschild (1981) look at a voting model where individuals act independently, the average individual votes with probability p, p is unknown and is distributed as $f(p)$. If we translate the decision to vote into one to tender, and denote by k the fraction of shares needed to ensure a successful bid, then the chance of being pivotal is approximately $f(k)/2N$ for large N. The reasoning is the once p is far away from k, the chance of success converges to either zero or one. To find the cases with a pivotal shareholder, we may ignore all of the distribution except the part closest to k. At $p = k$, the chance of being pivotal is approximately $1/\sqrt{2\pi k(1-k)N}$. This approximation holds for a range of p close enough to k so that the expected number of shares tendered falls within a standard deviation of k. The chance that p lies in this range is roughly $f(k)\sqrt{k(1-k)/N}$. Multiplying these together results in a probability proportional to $f(k)/N$. Even with the diminution of influence, down from $1/\sqrt{N}$ to $1/N$, similar arguments to those in Section 4 show that the raider can still capture a large fraction of the surplus when the number of shares per shareholder is large in relationship to N (as opposed to $\sqrt{N}$).

guarantees the offer will fail and gets nothing. Hence, all shareholders are willing to tender at any positive price.

While the limit result is intuitive, the fact that surplus always rises with K is more surprising. The reason is that when a greater fraction of shares is needed for success, this raises the chance each person is pivotal. A higher majority size corresponds to a greater probability of tendering, and this leads to less aggregate variance in the total number of shares tendered ($\sigma^2 = Np(1 - p)$). The chance of being pivotal is inversely proportional to the standard deviation of total shares tendered. That doesn't quite answer why $K/N = 99\%$ is better than $K/N = 1\%$, because both cases lead to low standard deviations. The difference is that the raider only gets the surplus when the pivotal shareholder tenders and the offer succeeds: This happens with chance p. Hence, expected surplus is proportional to $p/\sqrt{Np(1-p)}$, which is rising in p.[15] It only helps to make people pivotal when there is a high chance they will tender.

3.5 Conditional and Restricted Tender Offers

While unconditional tender offers can circumvent the free-rider problem, is this also true for conditional offers? It may appear that tendering is a weakly dominated strategy when the offer is conditional. Why accept $v < 1$ only in the circumstances when the raider succeeds, and the stock is worth 1? The incentive to tender under a conditional offer is entirely based on the probability that the offer will fail without you.

Proposition 2: *For any tender offer v' conditional on the raider gaining control, there exists an unconditional offer v that leads to an identical expected surplus for the raider and an identical probability of success. Similarly, for any unconditional offer v, there exists an equivalent conditional offer v'. This equivalence extends to restricted offers.*

Proof. Consider the decision problem for the typical shareholder facing an offer of v' conditional on the raider gaining control.

Tender	Don't Tender
$\frac{v'}{N}$ with chance $P(T_{-i} \geq K - 1)$	$\frac{1}{N}$ with chance $P(T_{-i} \geq K)$

In equilibrium, these expected rewards are equal: $v' = P(T_{-i} \geq K)/P(T_{-i} \geq K - 1)$. Thus, associated with each v' is a p. The raider's

15. This intuitive explanation depends on a normal approximation to the binomial. A combinatorial argument in the Appendix provides the formal proof.

expected surplus [eq. (2)] depends only on p. The same p can be chosen for the conditional and the unconditional offers: Let $v = v' * P(T_{-i} \geq K - 1)$. The only difference between the two offers is that the raider scales up v' by $1/P(T_{-i} \geq K)$ to compensate for the probability that those who tender will get nothing. For risk-neutral shareholders all the expected payoffs are the same as before. The case for restricted offers follows similarly, and is provided in the Appendix. □

In all three cases, the shareholder's expected surplus is equal to the value of free-riding, $P(T_{-i} \geq K - 1)$: this is *independent* of the type offer used. Thus, the residual surplus for the raider does not depend on the type of offer used.

The equivalence requires that the conditional tender offer be conditioned on the raider gaining control. In theory, a raider could demand an even greater percentage before agreeing to accept any shares. Thus, a conditional offer allows the raider to choose a super-majority rule. As we saw in Proposition 1, if he demands a 100% majority, he can capture all the surplus. Bagnoli and Lipman (1988) use this argument to claim the superiority of conditional offers. However, this type of conditioning may not be credible. If the tender price is less than 1, why would the raider refuse to exercise his option to buy the tendered stock provided he has control? Although the conditional bid does not *force* the raider to purchase the stock, it does not exclude him from doing so. Anytime he has enough stock to gain control, he should purchase all the tendered stock. Because shareholders can foresee this strategy, the only credible conditioning is on the raider gaining control.

There may be advantages to conditional offers that are not captured in the current model. If the raider is risk-averse, a conditional bid will lower the variance of his profits, and perhaps more important, guarantee that the raider never loses money. Financing considerations are also relevant. Often a bank lends a raider money using the target firm's assets as collateral. If the raider does not achieve control, he has no collateral, and the bank will not provide the financing. Hence, the raider is forced to make his bid conditional on receiving control.

3.6 The Raider as a Large Shareholder

Shleifer and Vishney (1986) demonstrate the empirical importance of large shareholdings in most major corporations. How does the raider owning a block R of shares affect his ability to engage in a takeover? Because he has control over his own shares, the raider need only capture $K - R$ shares out of the other $N - R$ shares. (All other shareholders have only one share.) Just as before, the raider's surplus on the shares he does not own is the chance of the tendering shareholders being pivotal; in addition, the raider gets all the expected increase in value on his own shares.

$$\text{Raider surplus} = \frac{N-R}{N}\binom{N-R-1}{K-R-1}p^{K-R}(1-p)^{N-K} + \frac{R}{N}\binom{N-R}{K-R}p^{K-R}(1-p)^{N-K}. \tag{6}$$

Raider surplus is proportional to $p^{K-R}(1-p)^{N-K}$. Differentiating with respect to p reveals that $p^* = K/[N-R]$.

Although this is the same formula as before, the implication is now very different. The expected number of shares tendered is $R + (N-R)p^* = R + K$. This is R shares more than necessary. Again if one thinks in terms of a normal approximation, the standard deviation for the total number of shares tendered is $\sqrt{(N-R)p^*(1-p^*)}$ and, thus, the raider has at least $(\frac{2R}{\sqrt{N-R}})$ standard deviations of safety. This suggests that once a raider has a large stake, he will choose a tender price to ensure that his bid is very likely to succeed. Most of the surplus on the shares he doesn't own goes to the present shareholders.

Example 2. Let $N = 10{,}000$, $K = 5{,}000$ (50% majority is needed for control) and $R = 100$ or 1%. Then $p^* = K/(N-R) \approx \frac{1}{2}$ and the number of standard deviations of safety for the raider is roughly $200/100 = 2$. With a one-tailed test, 2 standard deviations implies over a 97.5% chance of success for the raider.

This formulation gives us a handle on what it takes to be a "large" shareholder. For the raider to be big, $R/N > 1/\sqrt{N}$. In particular, the percentage needed to be big falls as the size of the company grows.

4. Shareholders with Many Shares

Sections 4.1 and 4.2 present the central results of the paper. We depart from the one-share-per-shareholder model to allow for large and potentially unequal holdings. An essential feature of the multiple-shares model is that it avoids the indivisibility problem. With only one share, tendering is all or nothing. When shareholders each have a large number of shares, it becomes possible to tender a fraction of holdings. In the limit, as shareholdings become infinitely divisible, the raider and the shareholders split the surplus. Shareholders are asked to tender half their holdings (for free), and in return they keep the rise in value on their remaining half.

The multiple shares model also allows us to consider the effect of unequal shareholdings. We show that concentration helps the raider. The biggest shareholders tender down to a common level, while small shareholders free ride. The concentration or asymmetry in shareholdings offers a way of coordinating shareholder tendering. Because large shareholders are expected to tender, small shareholders can free ride, which in turn reinforces the large shareholders' decision to tender.

Why are large shareholders the ones to tender? The explanation comes from a declining marginal incentive to tender. Tendering provides two benefits: one is the direct payment of v; the second is the increased chance of success. The second benefit is more highly valued, the more shares that are held. As the number of shares held diminishes, the increased chance of success is worth less and less. Once all shares are tendered, the former shareholder no longer cares about the outcome. The hardest share to motivate an individual to tender is his last one. This helps explain why the case where everyone has only one share minimizes the raider's surplus.

4.1 Finite Number of Shares

We are interested in the case where each shareholder owns a large number of shares so that shareholdings are divisible. We reach this case by letting the total number of shares increase through a stock split. Each initial share of stock becomes m shares of new stock (and each new share is entitled to $\frac{1}{m}$ of the initial dividend stream). This m-for-1 split increases the total number of shares without affecting a change in the distribution of ownership. We show that as m becomes large, the raider may capture a percent of surplus equal to the control majority—when a 50% majority is required for control, the raider captures half the surplus.

To demonstrate this result, it is useful to introduce additional notation. In the event that the original shares are split m for 1:

$S_i(m) = ms_i$ denotes the number of shares held by individual i,
$S(m) = m\Sigma_i s_i$ denotes the total number of shares,
$K(m) = [\delta S(m) + 1]$, where a δ-majority is needed for control,
$t_i(m) =$ shares tendered by i,
$T(m) = \Sigma t_i(m)$, the total number of shares tendered.

For simplicity of notation, we omit m as an argument in the following.

As in the previous sections, there are multiple Nash equilibria in response to a tender offer of v ($0 < v < 1$). Many of these are pure strategy equilibria in which exactly K shares are "designated" to be tendered, and each share tendered is pivotal. There is a mixed-strategy equilibrium that we believe is focal. The strategies are anonymous; the number of shares tendered depends only on the size of the shareholdings and not the identity of the person. The equilibrium has a symmetry property in that everyone tenders down to some common level (except those whose initial holdings are below this level, and they tender nothing). As we will see in the next section, this focal equilibrium is also the unique solution to the model where shareholders can tender a continuous fraction of their shares.

In the focal equilibrium, the large shareholders tender down to some common range z^* or $z^* + 1$.

$$z^* = \{\max z : \sum_i \max(S_i - z, 0) \geq K\}.$$

The definition of z^* is that it is the largest number such that if everyone keeps their initial holdings up to z^* shares (and tenders the rest), the takeover is still guaranteed to succeed.

Focal Equilibrium: *Let $I = \{i : S_i \geq z^* + 1\}$. For $i \in I$, a shareholder tenders $S_i - z^* - 1$ shares with certainty and tenders an additional share with chance p:*

$$\tilde{t}_i = \begin{cases} S_i - z^*, & \text{with probability } p; \\ S_i - z^* - 1, & \text{with probability } 1 - p. \end{cases}$$

For $i \in I^c$, no shares are tendered: $\tilde{t}_i = 0$.

Proposition 3: *Given any $v \in (0, 1)$, there is a p such that the strategies, $\tilde{t}_i$, are an equilibrium to the game.*

The equilibrium value of p is determined by

$$z^*P[T = K | t_i = S_i - z^*] + v = P[T \geq K | t_i = S_i - z^* - 1]. \tag{7}$$

To express this equation in words, the marginal gain from tendering equals the payment for tendering (v) plus the chance of increasing the value of all remaining shares when pivotal; this must be compared with the opportunity cost of not tendering, which equals the lost ability to free ride on the marginal share.

The surplus to the raider is the residual from the total surplus,

$$\begin{aligned} \text{Raider Surplus} &= P[T \geq K] - \sum_i t_i v + (S_i - t_i)P[T \geq K | t_i = t])/S \\ &\approx (P[T \geq K] - v) \sum_i t_i/S. \end{aligned} \tag{8}$$

The total of shares tendered will be approximately K (and always within N of K). Even as the number of shares becomes very large, the maximal variability in the number of shares tendered is N, as no individual randomizes over more than one share. Once m is large, $N/m \rightarrow 0$.

$$\text{As } m \rightarrow \infty, \text{ Raider Surplus} \rightarrow \frac{K}{S}\,(P[T \geq K] - v). \tag{9}$$

PROPOSITION 4: *For $v > 0$ and $K/S < 1$, as $m \rightarrow \infty$, the chance of a successful takeover approaches one, and the raider's expected surplus converges to $\frac{K}{S}(1 - v)$.*[16]

COROLLARY: *For $K/S < 1$, as $m \rightarrow \infty$, the raider can come arbitrarily close to capturing K/S of the surplus by letting v approach zero.*

When each shareholder has a large number of shares, the optimal tender offer is almost certain to succeed. A natural issue is the speed of convergence. How many shares must the average shareholder have in order for the chance of success to be greater than 90%? We show that the speed of convergence depends on the ratio of m to $\sqrt{N}$.

PROPOSITION 5: *As N and $m \rightarrow \infty$, the probability that the raider will succeed is most conveniently measured in terms of standard deviations. Let z represent the number of standard deviations by which the expected number of shares tendered exceeds the control majority K. For the case of equal shareholdings ($s_i = \frac{1}{N}$), z can be bounded by the inequality*

$$[1 - (K/S)]\,\frac{\sqrt{2}m}{\sqrt{\pi N}}\,e^{-z^2/2} \leq 1 - v.$$

COROLLARY: *Let $K/S = 50\%$ and $v = \frac{1}{2}$. Then $m/\sqrt{N} \geq 2$ implies that $z \geq 1.28$. The raider's probability of success is at least 90%, and correspondingly the raider's surplus is at least $(\frac{1}{2})(.9 - .5) = 20\%$. This ratio would be satisfied for a company with a 250,000 shareholders and 1,000 shares per shareholder.*

16. When $v \leq 0$, two cases are possible. One is the equilibrium described in Proposition 4. There is a second equilibrium where no one tenders, and the chance of success is 0. We focus on the case where v is strictly positive (but possibly small) for two reasons. Laws against manipulative tender offers below the current valuation create serious legal obstacles to completing a tender at less than the current market price. For negative v, the equilibrium where the success probability is zero seems more intuitive and this results in no surplus for the raider.

4.2 Shares Are Infinitely Divisible

An alternative way to present our argument is to consider the case where shares are continuously divisible, and there is some noise in the number of shares tendered. We prove that there is a unique equilibrium, and it corresponds to the focal equilibrium. As the amount of the random disturbance vanishes, the equilibrium converges to a pure-strategy solution. While there are a continuum of possible pure-strategy solutions in the limit case of zero noise, as the noise vanishes, the solution always converges to a particular pure-strategy equilibrium—the one in which the largest shareholders tender down to a common level.

With perfectly divisible shares, there is no randomness because of shareholders' mixing strategies. Thus, the uncertainty must be exogenous. We denote the randomness of the total mass of shares tendered by a continuous random variable ϵ, with mean 0, variance σ^2, and density $f(\epsilon)$.

We normalize the total number of shares to 1 and denote by γ_i the fraction of the firm held by i. If shareholder i tenders fraction λ_i of his holdings, the shareholder receives surplus

$$\text{Shareholder surplus} = \gamma_i[\lambda_i v + (1 - \lambda_i)P[T \geq K|\lambda_i \gamma_i, \ldots]] \tag{10}$$

Let $T = \bar{T} + \epsilon$, where the expected mass of shares tendered is denoted by

$$\bar{T} = \sum_i \lambda_i \gamma_i. \tag{11}$$

and ϵ represents the random disturbance.

The first-order condition (for individuals at an interior solution) is

$$v - P[T \geq K|\bar{T}] + (1 - \lambda_i)\gamma_i P[T = K|\bar{T}] = 0. \tag{12}$$

At a Nash equilibrium, $P[T \geq K|\bar{T}]$ and $P[T = K|\bar{T}]$ do not vary across the population. Thus, we can choose a C, constant across i, such that

$$C = \frac{(P[T \geq K] - v)}{P[T = K]}\,. \tag{13}$$

Maximization implies:

$$\begin{aligned} (1 - \lambda_i)\gamma_i &= C \quad & 0 \leq C \leq \gamma_i, \\ \lambda_i &= 1 \quad & C < 0, \\ \lambda_i &= 0 \quad & C > \gamma_i. \end{aligned} \tag{14}$$

As in the focal equilibrium, all shareholders with $\gamma_i > C$ tender down to a constant level. Those with holdings $\gamma_i \leq C$ tender nothing and free ride on the bigger shareholders.

This first-order condition approach will characterize a unique equilibrium if the maximization problem is strictly quasi-concave. Quasi-concavity follows whenever the upper tail of cumulative density of ϵ has an increasing hazard rate; in particular, this is satisfied for any normal density.

Assumption 1: *$ln[1 - F(\epsilon))]$ is concave.*

Proposition 6: *Under Assumption 1, there is a unique equilibrium.*

Given a unique solution, it is direct to characterize the raider's expected surplus:

$$\begin{aligned} \text{Raider Surplus} &= P(T \geq K) - \sum_i \gamma_i[\lambda_i v + (1 - \lambda_i)P(T \geq K)] \\ &= [P(T \geq K) - v] \sum_i \gamma_i \lambda_i \qquad (15) \\ &= [1 - F(K - \bar{T}) - v]\bar{T} \end{aligned}$$

By a choice of v near zero, the raider can capture a fraction of the surplus equal to the majority needed for control.

Proposition 7: *Under Assumption 1 and $v > 0$, as $\sigma^2 \to 0$, the raider's surplus converges to $(1 - v)K$.*

Note that if the raider is allowed to make negative tender offers, there are two possible equilibria. In one solution, he still captures $(1 - v)K$ of the surplus. In the second equilibrium, the offers fails with probability 1, and the raider gets no surplus. When the raider is offering a negative tender premium, the equilibrium with certain failure seems the more reasonable solution.

The infinitely divisible shares model is ideal for considering how a change in the distribution of shareholdings affects the raider's surplus. An increase in the concentration of holdings helps the raider. The worst case for the raider is a uniform distribution of shares among the shareholders.

Lemma 3: *Under Assumption 1, any change in the distribution of shareholdings that raises $\bar{T}$ raises the raider's expected surplus.*

Proof. Under Assumption 1, there exists a unique equilibrium. From eq. (15), the raider's expected surplus is $[1 - F(K - \bar{T}) - v]\bar{T}$. When $\bar{T}$ increases, both terms in the expression rise, so that the raider's expected surplus unambiguously rises. □

We say that shareholdings are more concentrated if the new distribution is the result of a stock transfer from some i to j where $\gamma_i \leq \gamma_j$. A change in the distribution of shares that takes away shares from those with small holdings and gives it to those with big holdings increases the raider's surplus.

PROPOSITION 8: *Under Assumption 1, an increase in the concentration of shareholdings results in (weakly) greater surplus to the raider.*

This result is similar in spirit to Theorem 5 in Bergstrom, Blume, and Varian (1986); these authors show that the Nash equilibrium level of contributions to the supply of a public good will be increased if there is a greater dispersion of income. In our model, the public good is tendering one's shares.

5. CONCLUSION

In challenging the results of Grossman and Hart, we have gone to the other extreme. The raider's potential surplus rises from zero to 50%. It seems equally implausible and counterfactual that raider captures either that much or that little.

Of course, our model is missing many important elements of the takeover process. In particular, we have only one bidder. It is frequently the case that hostile takeovers involve competition between several potential raiders. Once there is more than one bidder, no raider captures half the surplus. Before we try to develop a model with multiple raiders, it is important to first explain how takeovers are profitable when there is just one possible raider.

The point of this paper has been to reexamine the free-rider problem. We have focused on the two different ways to model the continuum: there may be a large number of shareholders and a large number of shares. The failure of the raider to capture any of the surplus depends critically on the assumption of equal and indivisible shareholdings—the one-share-per-shareholder model. In contrast, we have shown that once shareholdings are large and potentially unequal, a raider may capture a significant part of the increase in value. The free-rider problem does not prevent the takeover process when shareholdings are divisible.

6. APPENDIX

LEMMA 1: *In the one-share-per-shareholder model, all individuals who pursue a mixed strategy must tender with equal probability.*

Proof. For all individuals mixing, the chance of the raider succeeding without their share must be the same, $v = P(T_{-i} \geq K)$. If one of these shareholders, A, had a higher equilibrium chance of tendering than another of these shareholders, B, then A's absence would be more consequential than B's, $P(T_{-A} \geq K) < P(T_{-B} \geq K)$. That implies A has a *lower* value from free riding than B. In particular, this contradicts the fact that free riding must be equally valuable for all individuals who pursue a mixed strategy. □

LEMMA 2: *There exist* $\sum_{i=0}^{K-1} \sum_{j=K+1-i}^{N-i} \binom{N}{i}\binom{N-i}{j}$ *partial mixed-strategy equilibria. In a representative equilibrium,* i *shareholders tender with probability 1,* $N - i - j$ *shareholders tender with probability 0, and the remaining* j *pursue a mixed strategy.*

Proof. In any partial mixed-strategy equilibrium, the number of shareholders who tender with probability 1 can be any $i \leq K - 1$; otherwise, the offer will succeed with certainty, and everyone else will free ride. The minimum number of shareholders who participate in the mixed-strategy is the number j such that $i + j = K + 1$; otherwise the offer will certainly fail, or each person will be pivotal. The remaining $N - i - j$ shareholders free ride. The summation reflects the number of ways of dividing the N shareholders into these three groups.

The partial mixed-strategy equilibrium with i shareholders who tender, j who pursue a mixed-strategy, and $N - i - j$ who free ride involves the same mixing probabilities as for the case where there are only j shareholders (the value of the firm is j/N) and a majority size of $K - i$ is needed for control. When the three types coexist, each is still maximizing their utility. As was demonstrated in Lemma 1, a shareholder who has a greater chance of tendering must have a lower value of free riding. The shareholders "designated" to tender with $p = 1$ have the lowest value of free riding; they strictly prefer v to $P(T_{-i} \geq K)$. The shareholders designated to tender with $p = 0$ have the greatest incentive to free ride; they strictly prefer $P(T_{-i} \geq K)$ to v. Finally, the shareholders designated to pursue a mixed strategy are indifferent between v and $P(T_{-i} \geq K)$. □

PROPOSITION 1: *For* $K < N$, *raising the control majority increases the raider's surplus.*

Proof. With K raised to $K + 1$, the raider still has the option of choosing $p = K/N$, the p^* corresponding to control majority of K. Equation (A1) demonstrates that this lower bound on the raider surplus equals the previous maximal surplus with majority size K. Because maintaining a suboptimal p holds surplus constant, adjusting p optimally for $K + 1$ must lead to greater surplus level. Raider surplus is thus strictly increasing with the control majority.

$$\begin{aligned}\text{Raider Surplus}_{|(K+1,N)} &= \binom{N-1}{K} p^{K+1}(1-p)^{N-K-1},\ p = \frac{K+1}{N} \\ &> \binom{N-1}{K} \left(\frac{K}{N}\right)^{K+1} \left(\frac{N-K}{N}\right)^{N-K-1} \\ &= \binom{N-1}{K-1} \left(\frac{K}{N}\right)^{K} \left(\frac{N-K}{N}\right)^{N-K} \\ &= \text{Raider Surplus}_{|(K,N)} \qquad \square\end{aligned} \tag{A1}$$

PROPOSITION 2: *For any tender offer v' conditional on the raider gaining control, there exists an unconditional offer v that leads to an identical expected surplus for the raider and an identical probability of success. Similarly, for any unconditional offer v, there exists an equivalent conditional offer v'. This equivalence extends to restricted offers.*

Proof. Here we demonstrate the equivalence relationship extends to restricted tender offers. Consider a tender offer restricted to K shares. The payoff to tendering is

$$\frac{v}{N} + \frac{(1-v)}{N} \sum_{j=K}^{N} P(T_{-i} = j-1) \left[\frac{j-K}{j} \right].$$

The payoff to not tendering equals $\frac{1}{N}P(T_{-i} \geq K)$.[17] In the mixed-strategy solution, the two values are equal at the equilibrium p. Once again, the raider surplus depends only on p and in the same way as in eq. (2). Thus, with an appropriate change in v, the raider may achieve the same expected payoffs as with the other types of tender offers. $\square$

PROPOSITION 3: *Given any $v \in (0, 1)$, there is a p such that the strategies, $\bar{t}_i$, are an equilibrium to the game.*

Proof. The number of randomly tendered shares needed for success is $\tilde{K} = K - \sum_{i \in I}(S_i - z^* - 1)$. We represent the randomly tendered share by the variable x_i: $x_i = 1$ if i tenders the random share and $x_i = 0$ otherwise. The raider succeeds if and only if $X = \sum_{i \in I} x_i \geq \tilde{K}$. Let n be the cardinality of I. X is binomially distributed with parameters (n,p).

If $\tilde{K} = n$, set $p = 1$. All $i \in I$ are pivotal. Thus, each i will set $x_i = 1$; the tender offer succeeds with probability 1.

For $\tilde{K} \in [1, n-1]$, there is a symmetric mixed-strategy equilibrium. All $i \in I$ will be indifferent between tendering down to z^* and $z^* + 1$:

17. Note that one interesting feature of restricted offers is that we expect the stock price should rise above the tender price v. The reason is that there is a chance there will be a surplus of shares tendered, in which case the tenderer may have his share returned and receive the full value.

Tender $S_i - z^*$ shares	*Tender* $S_i - z^* - 1$ shares
$(S_i - z^*)v + z^*P[T \geq K \vert x_i = 1]$	$(S_i - z^* - 1)v + (z^* + 1)P[T \geq K \vert x_i = 0]$

Combining terms in z^* reveals

$$(z^* + 1)P[T = K|x_i = 1] + v - P[T \geq K|x_i = 1] = 0. \tag{A2}$$

At $p = 0$, this equation equals $z^* + v$ when $\bar{K} = 1$ and v otherwise: in both cases, the value is positive. When $p = 1$, the value falls to $v - 1 < 0$. Because the expression is continuous in p, a solution exists. Note that at this equilibrium value of p, for $i \in I^c$ the initial (and, hence, residual) shareholdings are less than $z^* + 1$ so that the value of eq. (A2) is negative.

It remains to argue that $\bar{t}_i$ is a globally optimal tendering strategy. This will follow if the reward for tendering t_i shares is quasi-concave in t_i. The marginal gain from tendering $t_i + 1$ over t_i may be rewritten as

$$P[T_{-i} \geq K - t_i - 1]\left(\frac{(S_i - t_i)P[T_{-i} = K - t_i - 1]}{P[T_{-i} \geq K - t_i - 1]} + \frac{v}{P[T_{-i} \geq K - t_i - 1]} - 1\right) \tag{A3}$$

We prove quasi-concavity by showing that eq. (A3) changes sign only once, from positive to negative. Consider the terms within the brackets as t_i increases. First, $(S_i - t_i)$ falls. Second, the hazard-rate factor, $\frac{P[T_{-i}=K-t_i-1]}{P[T_{-i}\geq K-t_i-1]}$ is reduced as the hazard-rate of the binomial distribution is increasing in K (and, hence, decreasing in t_i). Finally $P[T_{-i} \geq K - t_i - 1]$ increases with t_i so that $v/P[T_{-i} \geq K - t_i - 1]$ falls. Because each of the three effects are negative, the marginal gain from tendering changes sign at most once. □

Proposition 4: *For $v > 0$ and $K/S < 1$, as $m \to \infty$, the chance of a successful takeover approaches one and the raider's expected surplus converges to $\frac{K}{S}(1 - v)$.*

Proof. The probability of success must converge to one. If the probability of success was bounded by some $q < 1$, a shareholder in I could tender an additional N shares and guarantee the surplus on his remaining $z^* - N$ shares. As $m \to \infty$, the cost of this strategy is at most N/m, which converges to zero. But, the gain from guaranteeing success is bounded above zero:

$$(1 - q)(z^* - N)/m \to (1 - q)z^*/m \geq (1 - q)(1 - K/S) \sum_i s_i/N.$$

Because the probability of success converges to 1, the raider expects to capture the full increase in value on all the shares tendered, $\frac{K}{S}(1 - v)$.
□

PROPOSITION 5: *As N and $m \to \infty$, the probability that the raider will succeed is most conveniently measured in terms of standard deviations. Let z represent the number of standard deviations by which the expected number of shares tendered exceeds the control majority K. For the case of equal shareholdings ($s_i = 1/N$), z can be bounded by the inequality*

$$[1 - (K/S)] \frac{\sqrt{2}m}{\sqrt{\pi N}} e^{-z^2/2} \leq 1 - v.$$

Proof. We can measure the number of shares tendered in terms of standard deviations above the expected number needed for success. Because shareholdings are equal, all shareholders participate in the mixed-strategy. Given any value of $\bar{K}$, we can express p as $p = \bar{K}/N + a/\sqrt{N}$ for some a. The expected number of surplus shares tendered (or deficient if a is negative) is then $a\sqrt{N}$. Measured in terms of standard deviations, we have $a\sqrt{N}/[Np(1 - p)]^{\frac{1}{2}} > 2a$. Hence, if a is large, the probability of success will be close to 1. For example, $a = 1.96$ corresponds to a 97.5% chance of success (these are one-tailed confidence intervals).

To find a bound on a, we use (A2) and three facts:

1. $P[T \geq K|t_i = t] - v \leq (1 - v)$ as the probability of success is always less than 1;
2. $z^* \geq (1 - K/S)m$, by the definition of z^*;
3. In the limit as N and $m \to \infty$, $P[T = K|t_i = t + 1] \geq \frac{\sqrt{2}}{\sqrt{\pi N}} e^{-z^2/2}$ by the normal approximation to the binomial (where we have substituted the inequality $p(1 - p) \leq \frac{1}{4}$).

$$P[T \geq K|t_i = t] - v = z^*P[T = K|\ t_i = t + 1] \ \Rightarrow \qquad \text{(A4)}$$

$$(1 - v) \geq [1 - (K/S)]\ mP[T = K|t_i = t + 1] \Rightarrow$$

$$(1 - v) \geq [1 - (K/S)] \frac{\sqrt{2}m}{\sqrt{\pi N}} e^{-z^2/2}. \qquad \square$$

PROPOSITION 6: *Under Assumption 1, there is a unique equilibrium.*

Proof. The value of C in eq. (13) uniquely determines each shareholder's strategy. We first show that there exists a value of C for which the corresponding $\gamma_i(C)$ are optimal for each person. We then show this equilibrium is unique—there is only one value of $C \in [0, \max_i \gamma_i]$ that is consistent with optimization.

Combining eqs. (11) and (14) reveals that

$$\bar{T} = \sum_i \max((\gamma_i - C), 0), \qquad \text{(A5)}$$

so $\bar{T}$ is a declining function of C. Rewriting the first-order condition of eq. (12), in terms of $F(\epsilon)$ and $f(\epsilon)$ and then dividing by $[1 - F(K - \bar{T})]$ leads to

$$[1 - F(K - \bar{T})] \left\{ \frac{v}{[1 - F(K - \bar{T})]} - 1 + C \frac{f(K - \bar{T})}{[1 - F(K - \bar{T})]} \right\}. \quad \text{(A6)}$$

Note that the expression inside the brackets is strictly monotonic in C given the assumption that $\ln[1 - F(K - \bar{T})]$ is concave in $\bar{T}$ and the derivation that $d\bar{T}/dC < 0$. Thus, three (mutually exclusive) cases are possible: (1) the first-order condition is positive at $C = 0$ in which case $\lambda_i = 1$ and $\bar{T} = 1$; (2) the first-order condition is negative at $C = \max_i \gamma_i$ in which case $\lambda_i = 0$ and $\bar{T} = 0$; and (3) the first-order condition equals zero at some unique value of $C \in (0, \max_i \gamma_i)$ in which case $\lambda_i = \max[(1 - C/\gamma_i), 0]$. □

PROPOSITION 7: *Under Assumption 1 and $0 < K < 1$, as $\sigma^2 \to 0$, the raider's surplus converges to $(1 - v)K$.*

Proof. In any equilibrium, the chance of success, $1 - F(K - \bar{T})$, must be at least v or else tendering would be a dominant strategy. [If everyone tenders, then $1 - F(K - \bar{T})$ must be greater than v as $\sigma^2 \to 0$; see Lemma 4 later.] But as the density of ϵ becomes concentrated around zero, this requires that $\bar{T}$ approaches K. In the limiting case, any individual can guarantee that the offer succeeds with an arbitrarily small increase in λ_i. If the offer was not already succeeding with a probability approaching 1, this increase in λ_i would be optimal, contradicting the fact that an interior solution to the first-order conditions characterizes the unique best-response (Lemma 5 later). The chance of success converges to 1, providing the raider with expected surplus of $(1 - v)K$ as claimed. □

LEMMA 4: *Under Assumption 1, given $v < 1$ and $K < 1$, as $\sigma^2 \to 0$, $P(T \geq K|\bar{T} = 1) > v$.*

Proof. By Chebyshev's inequality $P(T \geq K|\bar{T} = 1) > 1 - \sigma^2/(1 - K)^2 \to 1$. □

LEMMA 5: *Under Assumption 1, for $v < 1$ and $0 < K < 1$, as $\sigma^2 \to 0$, $C \in (0, \max_i \gamma_i)$.*

Proof. If $C \leq 0$ then by eq. (14) all $\lambda_i = 1$ and consequently, $\bar{T} = 1$. Combining the definition of C in eq. (13) with Lemma 4 gives rise to a contradiction:

$$C \leq 0 \Rightarrow C = \frac{(P[T \geq K|\bar{T} = 1] - v)}{P[T = K|\bar{T} = 1]} > 0.$$

If $C = \max_i \gamma_i$ then $\lambda_i = 0$ and $P[T \geq K|\bar{T} = 0] < v$ as $\sigma^2 \to 0$, a contradiction. □

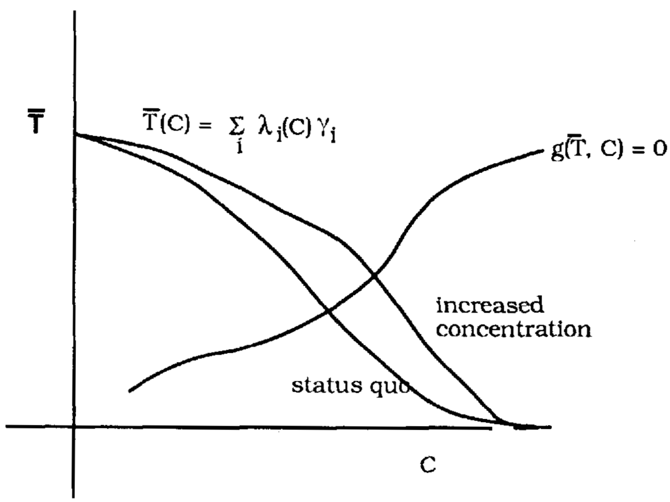
PROPOSITION 8: *Under Assumption 1, an increase in the concentration of shareholdings results in (weakly) greater surplus to the raider.*

Proof. Under Assumption 1, there exists a unique equilibrium. Let the original equilibrium value of C be C^*. If the transfer takes place exclusively among individuals with $\gamma < C^*$ and the transfers do not bring any individual above C^*, then there is no effect on the equilibrium. Similarly, there is no effect on the equilibrium if the transfers all take place for individuals with γ above C^*, and the transfers do not bring anyone below this level. In all other cases, the equilibrium is changed and the raider expects more surplus.

Holding C fixed, an increase in concentration increases $\bar{T}$ (by Lemma 6 later). Thus, the function $\bar{T}(C)$ is shifted outward, as indicated in the following figure. We can rewrite the solution to the first-order condition as a function of $\bar{T}$ and C, ignoring the interdependence:

$$g(\bar{T},C) = \frac{v}{[1 - F(K - \bar{T})]} - 1 + C\,\frac{f(K - \bar{T})}{[1 - F(K - \bar{T})]}\,.$$

From Proposition 6, we know that $dg/d\bar{T} < 0$ and it is direct that $dg/dC > 0$. Thus, the solution locus to $g(\bar{T},C) = 0$ is upward sloping, as indicated in the following figure. Hence, following an increase in concentration, in the new equilibrium, both C and $\bar{T}$ are (weakly) increased. This results in higher surplus to the raider by Lemma 3. □



LEMMA 6: *Under Assumption 1, an increase in concentration results in a (weakly) higher $\bar{T}$ for any positive value of C.*

Proof. Under Assumption 1, there exists a unique equilibrium. An increase in concentration is equivalent to a mean-preserving increase in the riskiness of the distribution of shareholdings. (The mean value of γ_i is $1/N$ by definition.) The expected number of shares tendered, $\bar{T}$, is a convex function of the distribution,

$$\bar{T} = \sum_i \max((\gamma_i - C, 0),$$

as max is a convex function. Thus, greater "riskiness" or concentration in the distribution of shareholdings leads to a (weakly) higher value of $\bar{T}$. □

REFERENCES

Bagnoli, M. and B. Lippman, 1987, "Successful Takeovers without Exclusion," *The Review of Financial Studies*, 1, 89–110.

Bebchuk, L., 1988, "Takeover Bids Below the Expected Value of Minority Shares," N.B.E.R. Working Paper #2524.

Bergstrom, T., L. Blume, and H. Varian, 1986, "On the Private Provision of Public Goods," *Journal of Public Economics*, 29, 25–49.

Chamberlain, G. and M. Rothschild, 1981, "A Note on the Probability of Casting a Decisive Vote," *Journal of Economic Theory*, 25, 152–162.

Grossman, S. and O. Hart, 1980, "Takeover Bids, the Free Rider Problem, and the Theory of the Corporation," *Bell Journal of Economics*, 11, 42–64.

Shleifer, A. and R. Vishney, 1986, "Large Shareholders and Corporate Control," *Journal of Political Economy*, 94, 461–488.