MULTI-DIMENSIONAL PRODUCT DIFFERENTIATION AND PRICE COMPETITION

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1. Introduction

BEGINNING with Hotelling's (1929) study of spatial competition, location theory has provided important insights into imperfect competition. However, the applicability of the classical approach is limited; it leaves two major issues unresolved. First, what impact does price competition have on the process of product selection? Second, what results are available when products differ across more than one dimension? We develop a simple framework to study competition through product differentiation and prices in a multi-dimensional setting.

Efforts to model price competition, even in a one dimensional setting, have proved discouraging. Equilibrium may not exist if firms locate too close together (d'Aspremont *et al.*, 1979). When equilibrium exists, its nature depends heavily on an exogenously specified transport cost function, be it fixed, linear, or quadratic (Gabszewicz and Thisse, 1985; Economides, 1985). In general, it is hard to provide an economic interpretation and justification for these types of transportation costs in a model of product differentiation.

Most location models follow Hotelling in focussing on competition along a single dimension. In one dimension, a firm has at most two neighbors; this places an artificial limit on the extent of competition. Only in higher dimensions is this problem eliminated (Archibald and Rosenbluth, 1975; Stiglitz, 1984). Realistically, product differentiation is almost always multi-dimensional. Computer printers vary in terms of speed, noise, and clarity of output. Cars vary in size, comfort, sportiness, fuel economy, reliability, and in many other dimensions.

The multi-dimensional nature of product design is particularly evident in marketing new products. Products are frequently launched to fill a perceived market niche; low alcohol beer and caffeine free soft drinks are two recent examples. These products take advantage of a characteristic under-exploited by existing competitors. This suggests that a multi-dimensional setting is important both to the theory of product design and to the theory of advertising. However, to understand product design first requires an understanding of price competition.

Product selection and pricing strategy is typically a two-stage process. Initially, product characteristics are chosen. Then given a set of commodities with fixed characteristics, competition takes place through prices.

Multi-dimensional product differentiation and price competition are brought together in our study of imperfect competition. We use Grandmont's

(1978) theory of intermediate preferences to build a tractable multi-dimensional location model. This provides a framework to address important issues in industrial organization: when are there first-mover advantages in product design (Guasch and Weiss, 1980); how is location chosen to deter entry (Schmalansee, 1978); what are the signs of collusive behaviour in product design; how do cost differences affect product differentiation, prices, and market shares?

If location choice is followed by price competition Proposition 4.1 provides conditions for existence of a price equilibrium given any location pair. The significance of this result stems from the historic difficulty in attaining existence of price equilibria in location models. In addition, it allows us to examine the effects of *cost asymmetries* on product differentiation. Proposition 4.2 shows that a low cost entrant seeking to capture the entire market from a higher cost incumbent does best to imitate his product. We also show that a high cost firm, by locating well away from the center of the market, may be able to avoid destructive competition with its low cost rival, and guarantee itself a market niche.

We present more precise results for product differentiation in a price-regulated duopoly. The first firm to fix a location is at a disadvantage. The second mover advantage increases with the dimension of the product space. For arbitrary distributions of consumer preferences over n-dimensional goods ($n \geq 2$), the second firm can capture up to $(n-1)/n$ of the market. However, when there is a measure of agreement (defined in Section 5) about which mix of characteristics is most desirable, the incumbent firm can always locate so as to guarantee a 36.8% $(1/e)$ market share.

Section 2 presents a brief survey of location theory. Section 3 outlines our general model. Section 4 integrates product design with endogenous price competition in a two-dimensional setting. Section 5 analyzes product differentiation in arbitrary dimensions for price regulated duopolists. Section 6 offers our conclusions.

2. Location theory

Location theory begins with the work of Hotelling (1929). In his analysis, the set of possible locations for a given product is represented by the unit interval, [0,1]. Each firm produces at a single location. Market demand is generated by consumers who differ in their location along a line. Each consumer prefers products closer to their own location. Prices are taken as fixed and identical and hence consumers always buy the closest product. This basic model has been applied by political scientists to study the choice of party platforms (Black, 1948; Kramer, 1977), and by economists to examine product differentiation (Chamberlin, 1933; Eaton and Lipsey, 1975; Prescott and Visscher, 1977; Lane, 1980).

Attempts to incorporate prices have encountered a major difficulty in even proving the existence of equilibrium strategies. It is largely this difficulty which has prevented the more widespread adoption of the model. To compute the equilibria for a two-stage model requires the existence of Nash equilibria in prices for *arbitrarily* close product specifications. The disturbing result of d'Aspremont *et al.* (1979) is that no pure strategy pricing equilibrium may exist when firms are located too close to one another. But, if no price equilibrium exists for certain locational choices then there is no way for firms to estimate the profitability of those locations.

Resolutions of the existence problem have concentrated on three areas: changing the transport cost function, allowing for mixed strategies, and focussing on vertical as opposed to horizontal location problems.

Gabszewicz and Thisse (1986) have demonstrated the existence of a pure strategy price equilibrium for the Hotelling model when there are quadratic as opposed to linear transportation costs. The equilibrium locations are at the two ends of the unit interval. Similar are the results of Economides (1983) who shows that when consumers have a maximal or reservation distance that a pricing equilibrium exists and firms locate far apart. The equilibrium depends heavily on the form of the transportation cost function (Gabszewicz and Thisse, 1986). As a consequence, few applications have been developed.

Even when there is no equilibrium in pure strategies, there may be mixed strategy solutions (Dasgupta and Maskin, 1986). It is of interest that Hotelling's model with linear transportation costs and bounded reservation prices possesses no equilibrium even in mixed strategies. In games where mixed strategy equilibria do exist (see examples by Osborne and Pitchik (1982) and Stiglitz (1984)), their complexity effectively rules out comparative static analysis.

Shaked and Sutton (1982) attain important positive results in price/location theory by examining vertical rather than horizontal product differentiation. In their model, firms compete over quality and price. Quality choice is a "vertical" location problem because all consumers prefer higher quality to lower quality. By contrast, in "horizontal" location problems, changing the product specification is a move towards some consumers and away from others. Their results are encouraging, but so far limited to problems of one dimension (quality). It has proved difficult to develop generalizations that include horizontal product differentiation.

An entirely different approach to multi-dimensional product differentiation and pricing is taken by Spence (1976), Dixit-Stiglitz (1977). They address Chamberlin's question of whether the competitive market will provide the optimal amount of product diversity. To focus on this question, they use a completely symmetric model. This bypasses issues of product design and questions where asymmetries play a prominent role.

To achieve greater applicability of the location model to multi-

dimensional horizontal product differentiation with price competition, we begin by reformulating the basic framework.

3. The general model

The formal model provides a precise specification of the consumer and producer sides of the economy and the definition of equilibrium. After presenting the general framework, we specialize to the case of duopolistic competition.

Consumers

Products are identified by an n-dimensional vector of salient characteristics (Lancaster, 1966; Gorman, 1980). Consumers face a variety of commodities each offering a different bundle of characteristics. There is exclusivity in consumption; each consumer chooses only one of the products. The characteristics of the commodities cannot be combined to yield intermediate mixes.

Our approach to the price/location model separates the space of consumer preferences from the space of product characteristics. The population is represented by a distribution on the space of utility functions. Specifically, we assume that consumers have Cobb–Douglas utility functions. Heterogeneity of tastes is then specified by a distribution function over the Cobb–Douglas parameters, as in Lane (1980).

In the traditional location model, a single space is traditionally used to represent both the diversity of products and the diversity of consumer preferences. A product is represented as a point in characteristic space. A consumer is represented by a most preferred characteristic mix and by a "transport cost" function. This function specifies the price difference needed for the consumer to be indifferent between a unit of any given good and a unit of their most preferred good. The model of consumer preferences underlying the transport cost function is left unspecified. A conceptual advantage of our approach is that "transport costs" are defined endogenously from the utility function rather than being exogenously set as either fixed, linear or quadratic.

An additional motivation for assuming Cobb–Douglas utility functions is tractability. There should be a simple calculation determining which consumers prefer good X to Y and which prefer Y to X. In one dimensional problems, this is straightforward. But, even in the two-dimensional location model with linear transportation costs, market areas are defined by hyperbolas when the two firms charge different prices. In contrast, when consumer preferences are derived from Cobb–Douglas utility functions, for all prices (P_x, P_y) the set of consumers who prefer X to Y is divided from those who prefer Y to X by a hyperplane in the space of Cobb–Douglas parameters [Grandmont (1978)].

Producers

On the supply side, there is a fixed list of potential competitors, $i = (1, \ldots, j)$. We assume that the order of entry into the market is sequential, with a fixed known order of entry. Based on the expected competition, the ith firm is either deterred from entering or pays market entry costs of A^i and produces one product. If it enters, competitor i has cost function $C_i(W)$ giving its unit cost of production for any list of product characteristics, W. Once chosen, product characteristics are fixed. Behind this assumption is the belief that changing product design is both time consuming and costly relative to changing price.

Each potential entrant decides on its location rationally, taking account of the existing products, future product selection, and the anticipated price competition. After all product locations are chosen, the prices are determined in a Bertrand–Nash equilibrium for the given products.

Equilibrium

An equilibrium of the game is an ordered list of j firms $\{f_1, f_2, \ldots, f_j\}$, j products $\{W_1, W_2, \ldots, W_j\}$, and j prices $\{p_1, p_2, \ldots, p_j\}$ where

(i) $\{p_1, p_2, \ldots, p_j\}$ is a Nash equilibrium in prices given products $\{W_1, W_2, \ldots, W_j\}$.

(ii) Each firm in the list $\{f_1, f_2, \ldots, f_j\}$ picks its location optimally in relation to other firms' prior locations, predicted future locations, and the ensuing price competition. Each firm that locates correctly anticipates making sufficient profits to justify entry.

(iii) Each non-entering firm rationally predicts a loss should it choose to enter.

Duopoly

In a duopoly, the incumbent firm first chooses its product, $X \in \mathbb{R}^n$. The rival firm responds with its product $Y \in \mathbb{R}^n$. Firm X has cost function $C_x(X)$, firm Y has cost function $C_y(Y)$. Entry into the market is free.

In this n-dimensional setting, a consumer of type $\alpha = (\alpha_1, \ldots, \alpha_n)$ has Cobb–Douglas utility function,

$$U(\alpha, Q, w_1, \ldots, w_n) = Q^b w_1^{\alpha_1} \ldots w_n^{\alpha_n}, \tag{1}$$

where w_i represents the ith characteristic of commodity W and Q represents the total consumption of other products. Consumption of any given commodity is treated as a continuous variable.

For simplicity, all consumers are assumed to have identical values of $b \geq 0$, and equal incomes $I > 0$. We normalize so that $\alpha \in S^n$, the n-dimensional unit simplex, i.e. $\alpha_i \geq 0$ and $\sum_{i=1}^{n} \alpha_i = 1$. Thus all individuals

spend the same amount, $I/(1+b)$, on commodity X or Y, whichever they purchase.[1] Units are chosen so that $I/(1+b)=1$.

Given commodities X and Y, the duopolists simultaneously choose prices P_x and P_y. A consumer with Cobb–Douglas parameter $\alpha \in S^n$ will purchase X over Y if and only if,

$$\frac{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{P_x} > \frac{y_1^{\alpha_1} \cdots y_n^{\alpha_n}}{P_y} \tag{2}$$

Taking logarithms and rearranging yields

$$\sum_{i=1}^{n-1} \alpha_i \ln\left(\frac{x_i y_n}{y_i x_n}\right) > \log\left(\frac{y_n}{x_n}\right) + \log\left(\frac{P_x}{P_y}\right) \tag{3}$$

Equation (3) asserts that the set of consumers preferring X to Y can be separated from those preferring Y to X by a hyperplane in the space of Cobb–Douglas parameters. This observation follows closely from Grandmont's (1978) demonstration that Cobb–Douglas utility functions lie in the class of intermediate preferences. The dividing hyperplane has normal vector

$$\left(\log\left(\frac{x_1 y_n}{y_1 x_n}\right), \ldots, \log\left(\frac{x_{n-1} y_n}{y_{n-1} x_n}\right)\right),$$

which depends only on the products' specifications, not on their prices. Hence price changes cause parallel shifts in the hyperplane dividing the population between the two products. The central role of hyperplanes greatly simplifies our analysis. This is most evident in Section 5, which presents a multi-dimensional extension of the fix-price models of Hotelling (1929) and Prescott and Visscher (1977).

4. Existence of a price equilibrium

We focus on a two-dimensional version of the general duopoly model. We assume that both firms have constant returns to scale and that production of either characteristic is equally costly. This allows us to consider the effect of cost differences between the two firms. Since the goods are infinitely divisible, no generality is lost in choosing a scale so that $x_1 + x_2 = y_1 + y_2 = 1$. Unit costs for Firm X are C_x and for Firm Y are C_y.

In two dimensions the determination of market shares is considerably simplified. Consumers buy (X, P_x) over (Y, P_y) if $\alpha_1 \geq \alpha_1^*$, where

$$\alpha_1^* = [\log(y_2/x_2) + \log(P_x/P_y)]/K, \tag{3'}$$

and $K \equiv \ln[(x_1 y_2)/(y_1 x_2)]$.[2]

[1] If expenditure differed across the population, an additional integration would be required holding $\sum_{i=1}^{n} \alpha_i$ constant.

[2] When locations are fixed, we assume that X is more intensive in the first characteristic so that $k > 0$. If $k < 0$, then market areas are reversed.

Without loss of generality, total population is 1. Firm X's revenue is then $F(\alpha_1^*)$, where $F(\alpha_1^*)$ is the proportion of the population with $\alpha_1 \leq \alpha_1^*$.[3] Consumers with Cobb–Douglas preferences who purchase X spend a fixed amount on X regardless of its price. Higher prices lead to lower production costs as each consumer's demand is inversely proportional to price. Thus profits obey

$$\begin{aligned} \Pi_x &= [1 - F(\alpha_1^*)][1 - C_x/P_x], \\ \Pi_y &= F(\alpha_1^*)[1 - C_y/P_y]. \end{aligned} \tag{4}$$

The major result of this section is an existence result for a pure strategy Nash equilibrium in prices given any two distinct commodities (X, Y). Existence depends on an assumption about the distribution of consumer preferences.

Assumption A1: The distribution function $F(\alpha_1)$ is concave and twice differentiable over its support (a, b); $F'(\alpha_1) = f(\alpha_1)$, $F''(\alpha_1) = f'(\alpha_1) \leq 0$.

One special case of A1 is a uniform density over a connected subset of the unit interval.

The point of this assumption is to prevent extreme distributions of consumer preferences. Profit is guaranteed to be well behaved if a sufficient fraction of the population is "centrally" located in relation to the consumers with "extreme" tastes. An equivalent assumption plays a central role in providing stability to a broad class of voting problems (see Caplin and Nalebuff, 1986).

To review, the order of events is that Firm X first chooses its characteristic mix (x_1, x_2) and then Firm Y selects (y_1, y_2). The prices (P_x, P_y) are subsequently determined as a Bertrand–Nash equilibrium for the given pair of products. To solve the game, it is necessary to work backwards and calculate the expected price equilibrium for any pair of locations.

Given locations and Firm Y's price, the derivative of Firm X's profit function must be zero in equilibrium:

$$d\Pi_x/dP_x = \frac{[1 - F(\alpha_1^*)]C_x}{P_x^2} - \frac{f(\alpha_1^*)(1 - C_x/P_x)}{KP_x} = 0, \tag{5}$$

where we have substituted in the value of $d\alpha_1^*/dP_x$,

$$d\alpha_1^*/dP_x = 1/[KP_x]. \tag{6}$$

Parallel calculations determine the derivative conditions for Firm Y,

$$d\Pi_y/dP_y = \frac{F(\alpha_1^*)C_y}{P_y^2} - \frac{f(\alpha_1^*)(1 - C_y/P_y)}{KP_y} = 0. \tag{7}$$

[3] We assume that the density of F is non-atomic so that consumers indifferent between distinct goods X and Y are negligible.

Proposition 4.1 demonstrates that the first-order conditions, (5) and (7) are also *sufficient* to identify Bertrand–Nash equilibria. This is the key step in our proof of existence of an equilibrium.

Proposition 4.1: For consumer preferences satisfying A1, any joint solution to the first-order conditions is a Bertrand–Nash equilibrium.

Proof

In Appendix. □

Proposition 4.2 shows that there exists a joint solution to the first-order conditions and that it is unique. In equilibrium, $d\Pi_x/dP_x = 0$ and $d\Pi_y/dP_y = 0$ simultaneously. This implies:

$$\begin{aligned} P_x/C_x &= 1 + K[1 - F(\alpha_1^*)]/f(\alpha_1^*), \\ P_y/C_y &= 1 + KF(\alpha_1^*)/f(\alpha_1^*). \end{aligned} \tag{8}$$

Taking ratios and substituting in the value of α_1^* in terms of P_x/P_y from equation (3′) yields the following implicit equation for any equilibrium value of α_1^* interior to the support of F,

$$\psi(\alpha_1^*) \equiv K\alpha_1^* - \log\left[\left(\frac{C_y}{C_x}\right)\left(\frac{f(\alpha_1^*) + K[1 - F(\alpha_1^*)]}{f(\alpha_1^*) + KF(\alpha_1^*)}\right)\left(\frac{y_2}{x_2}\right)\right] = 0. \tag{9}$$

Proposition 4.2: Under Assumption A1, there exists a unique equilibrium price pair (P_x, P_y) given locations (X, Y).

Proof

If X and Y are identical, then $P_x = P_y = \text{Max}[C_x, C_y]$ and the lower cost firm monopolizes the market. For X and Y distinct, Lemma A1 in the appendix, demonstrates that ψ is a strictly monotonic function for α_1 in the support of F. Hence, any solution to $\psi(\alpha_1^*) = 0$ is unique. If $\psi(\alpha_1)$ is everywhere negative then $F(\alpha_1^*) = 1$, $P_x = C_x$, and $P_y = C_x[1 + K/f(\alpha_1^*)]$. If $\psi(\alpha_1)$ is everywhere positive, then $F(\alpha_1^*) = 0$, $P_y = C_y$, and $P_x = C_y[1 + K/f(\alpha_1^*)]$. If a solution to $\psi(\alpha_1) = 0$ exists, then equilibrium prices are determined by substitution back into equation (8) □

Consider Firm Y's location decision in light of Proposition 4.2. It faces an incumbent of known location. If Firm Y has lower costs, it faces a dichotomous choice: it must decide between choosing a location in order to monopolize the market and choosing a location which results in a shared market.[4] Proposition 4.3 characterizes the optimal exclusionary tactic. There is an interesting analogy here with Hotelling's principle of minimum differentiation—the optimal exclusionary tactic is to locate in the identical

[4] On this point, Lane (1980) makes an a priori assumption that exclusion will never occur. Yet, to calculate equilibrium locations, one must allow for exclusion out of equilibrium.

spot as the incumbent. However, the motivation for imitation is entirely different than in Hotelling. Rather than agreeably splitting the market, the new entrant steals the entire market.

Proposition 4.3: When a low cost entrant (Firm Y) faces a high cost incumbent (Firm X) located in $[a, b]$, its optimal exclusionary strategy is to produce an identical product.

Proof

Consider all equilibria to the pricing game where Firm Y captures the entire market, $F(\alpha_1^*) = 1$. Since Firm X's profits in all of these equilibria are zero, it follows that $P_y \leq C_x$. If not, Firm X could earn positive profits by charging P_y and selling to consumers who prefer its product. Hence, over all positions (X, Y) such that $F(\alpha_1^*) = 1$, the highest value of P_y is C_x. Thus

$$\Pi_y \leq 1 - C_y/C_x. \tag{10}$$

But by locating in the same place as X and charging C_x, Firm Y manages to achieve a monopoly position with a profit of

$$\Pi_y = 1 - C_y/C_x, \tag{11}$$

proving the proposition □

Note that if the high cost incumbent is located outside the support of F then the optimal exclusionary tactic is to locate at the nearest boundary, a or b.

The alternative to exclusion is accommodation. Whether Firm Y chooses to exclude or accommodate depends on the density function and Firm X's initial location. If the density is continuous at both boundaries of the support, $f(a) = f(b) = 0$, then Firm X is guaranteed a positive equilibrium market share and hence positive profits for any initial location. This follows from the first-order conditions. If Firm Y has the entire market, it will wish to raise its price since the density of marginal consumers is zero. This allows Firm X to gain a positive market share.

With $f(a)$, $f(b)$ strictly postive, a high cost firm may be excluded from the market. For example, if the low cost firm locates first, then there is no guarantee that the high cost firm can fit into the market. Alternatively there may be uncertainty about the competitor's costs. In this case, even if the high cost firm locates first, it is not sure how much of an extreme position to take. There will then be a tradeoff between lower profits at more extreme positions and a higher probability of being excluded.

5. Multi-dimensional product differentiation without price competition

Consider competition in product design in an industry with regulated pricing. To focus on product differentiation, we assume that the cost functions, in addition to being linear and additive in characteristics, are

identical for the two firms X and Y. With an appropriate choice of units, cost are

$$C_x(w_1, \ldots, w_n) = C_y(w_1, \ldots, w_n) = \sum_{i=1}^{n} w_i.$$

Prices are a fixed mark-up over input costs $P_x/C_x = P_y/C_y = 1 + m$, for $m \geq 0$. With these assumptions, any two goods X and Y both in S^n will have an equal price.

With mark-ups fixed, total industry profits obey,

$$\Pi_x + \Pi_y = \left[1 - F(\alpha_1^*)\right]\left(1 - \frac{C_x}{P_x}\right) + F(\alpha_1^*)\left(1 - \frac{C_y}{P_y}\right)$$

$$= \frac{m}{1+m}.$$

Hence total profits are independent of product choice, and the two firms are playing a zero-sum game. They choose product design to maximize their own market share, or equivalently to minimize the competitors market share.

There is a simple correspondence between product choice and points in the space of Cobb–Douglas parameters. For a consumer of type $\alpha \in S^n$, the product $X = \alpha$ is most preferred among the possible equally priced products. This allows us to speak of product selection in the space of utility parameters. A product of type $\alpha \in S^n$ represents the most preferred point for the α type.

According to equation (3) of Section 3, the population is divided between products of type α_1 and α_2 by a hyperplane containing α_1, on one side, and α_2 on the other. The goal of Firm X is to produce for a type α_x in order to maximize its market share. It must anticipate that Firm Y's product α_y will depend on α_x: α_y will be chosen to minimize Firm X's market share. In equilibrium, Firm X chooses that type α^* which minimizes Firm Y's maximal market share.

Facing a given product α_x, Firm Y's maximization problem is straightforward. It picks α_y so that the hyperplane in the space of Cobb–Douglas parameters dividing the population between the two products contains maximal area on the side of α_y. The central observation is Lemma 5.1. Firm Y can, by an appropriate choice of product, secure for itself the population on *either* side of *any* hyperplane through α_x.

Lemma 5.1: Consider any given hyperplane in $\mathbb{R}^n$ which passes through the origin, and has a non-empty intersection with the unit simplex, S^n. For any product $X \in S^n$ which does not lie on this hyperplane, there exists a distinct product $Y \in S^n$ so that the given hyperplane separates those who prefer X from those who prefer Y.

Proof

A hyperplane is uniquely defined by a point on its surface and its normal. Given any normal vector $\pi = (\pi_1, \ldots, \pi_n)$ and a commodity $X \in S^n$, we construct a good $Y \in S^n$ so that the separating hyperplane has normal π and passes through the origin.

Let the ith characteristic of Y be $y_i = x_i e^{-\lambda \pi_i}$. Consumers indifferent between X and Y satisfy $\alpha \cdot \pi = 0$. Thus, for any non-zero λ, the separating hyperplane passes through the origin and has normal vector π. The scale λ is chosen so that $Y \in S^n$. □

With Lemma 5.1, the problem is formally identical to existing models in the social choice literature (see for example Grandmont (1978), Greenberg (1979) and Kramer (1977)). The optimal policy for Firm X is to produce for the type α^* that minimizes the maximal proportion of the population on one side of any hyperplane passing through α^*.[5]

We exploit the analogy with social choice to present some examples and some general results on equilibrium market shares.[6] The results explore the relationship between dimensionality, population preferences, and the order of entry into the market.

The case with $n = 2$ is straightforward with a non-atomic distribution of parameters. Two firms locating sequentially will share the market evenly. To secure a fifty percent market share, Firm X produces the product α^* which is most preferred by the median type, $F(\alpha^*) = \frac{1}{2}$. The political analogue is the well-known median voter result for a population with single-peaked preferences along a line (Black [1948]).

The simplicity of the two-dimensional case is misleading. Equal division does not generalize to higher dimensions, except in the special case where the distribution of Cobb–Douglas parameters is radially symmetric (Plott, 1967).

A more interesting example involves Cobb–Douglas parameters distributed uniformly in the three-dimensional unit simplex. Whenever the first firm locates, the second can receive more than one-half of the market. In fact it is optimal for the Firm X to produce the good $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ designed for the individual at the center of mass of the simplex.[7] At the center of mass, Firm X serves four-ninths of the population when the second firm locates optimally [Fig. 1(a)]. If Firm X locates elsewhere, Firm Y gets a greater than five-ninths share of the market [Fig. 1(b)].

In three dimensions, the worst possibility for Firm X involves an atomic

[5] In fact Lemma 5.1 shows only that Firm Y can achieve market areas arbitrarily close to those defined by hyperplanes through α_x. The epsilon-equilibrium is the relevant solution concept.

[6] Caplin and Nalebuff (1986) present a thorough treatment of the equivalent problem in the context of social choice.

[7] It is tempting to conjecture that the first firm always locates at the center of mass of population preferences even with more complex densities. This is false (Caplin and Nalebuff, 1986).

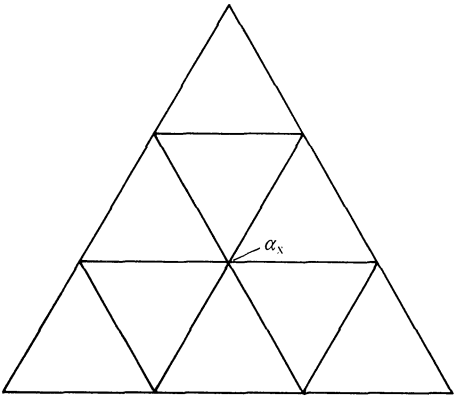

FIG. 1(a). The lines inside the triangle divide it into nine identical triangles. With Firm X locating at α_x, Firm Y can secure five of the nine triangles. To do this it produces products α_Y for which the dividing lines approach α_x at an angle perpendicular to any side of the outer triangle. There are no other lines in the triangle which secure as much as five-ninths of the population for Firm Y.

distribution of parameters. Assume that one third of the population is of type α_A, one third is of type α_B, and one third of type α_C, with α_A, α_B and α_C non-colinear. In this case, Firm Y can secure a two-thirds market share regardless of Firm X's location (Figure 2).

Insights from these planar examples are central to understanding higher-dimensional problems. There are two main results, both drawn from the social choice literature.

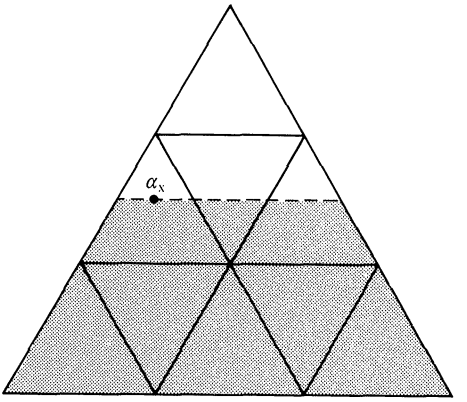

FIG. 1(b). Here, Firm X locates at α_x away from the centroid of the simples. Firm Y can now get more than five-ninths of the triangle. For example, it can pick α_Y so that the dividing lines approach α_X at an angle perpendicular to the farthest side.

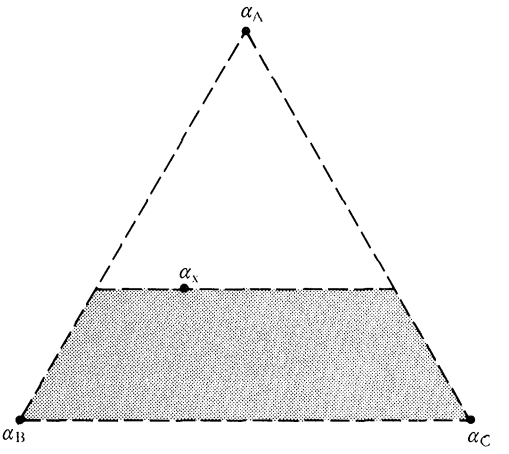

FIG. 2. The market consists of equal numbers of the three types α_A, α_B, α_C. Firm X locates at α_x. To secure the two types α_B and α_C, Firm Y produces a product giving rise to the dividing line drawn in the figure.

Proposition 5.1: With Cobb–Douglas parameters distributed arbitrarily over $S^n, n \geq 2$, Firm X locating optimally will receive between $\frac{1}{2}$ and $1/n$ of the total market. No tighter bound is possible.

Proof

This follows from the result of Greenberg (1979) in light of Lemma 5.1 □

Proposition 5.2: Consider Cobb–Douglas parameters distributed according to a non-atomic density $f(\alpha)$ on $\alpha \in S^n, n \geq 2$. Assume that the density $f(\alpha)$ is concave on its support $B \subset S^n$. Then Firm X locating optimally will receive a market share of between $\frac{1}{2}$ and $((n-1)/n)^{n-1}$. No tighter bound is possible.

Proof

This follows from the result of Caplin and Nalebuff (1986) in light of Lemma 5.1 □

These results highlight the importance of the dimensionality of the commodity space. In both propositions, the first firm's market share falls as the dimensionality rises. This suggests that entrants can improve their market share by differentiating their product along a previously unexploited dimension. Advertising is one tool used to expand the number of a product's salient characteristics. Burger King's advertising campaign introduce the distinction between flame-broiled and fried hamburgers. The recently introduced low-alcohol beer, LA, added a new dimension to beer marketing.

6. Concluding remarks

Our concluding remarks offer three directions for further research on multi-dimensional product differentiation and price competition.

Further development of the basic duopoly model

The solution to the simple duopoly model can be applied to examine a wide variety of other questions. Two important issues are the introduction of uncertainty into the locational model, and a comparison with the collusive solution.

The introduction of uncertainty will greatly enhance the model's realism. Typically, firms are not fully informed about competitors' costs. This informational asymmetry introduces entirely new issues. For example, the choice of a particular product may act as a signal of the firm's costs. This has implications both for an incumbent's choice of an initial location and the nature of the subsequent price competition. Here it is of interest that a central location by the first firm appears to be a high risk strategy once one takes account of uncertainty. Taking a central location is good for a low cost producer, but is bad should the rival turn out to have lower costs. Hence increased uncertainty about the other firm's cost may push both away from the center of the market. In equilibrium, locating centrally would send a strong entry-deterring signal that costs are very low.

The duopoly model can also be applied to study anti-trust questions. Collusion can take place in pricing decisions and/or location choices. Traditionally, anti-trust focus has been on recognizing price collusion rather than collusion in product design. Less obvious are instances where firms collude in product design, anticipating non-cooperative pricing strategies. For example, in the case of airlines, given that they pursue competitive pricing policies, are they colluding or competing in scheduling their flights?

Extensions

While the two firm, two-dimensional version of the model is revealing it is important to extend the framework to more realistic environments. Existence of a price equilibrium for an arbitrary pair of locations can be generalized beyond the two-dimensional existence result in Section 4. Our assumption of concavity (A1) plays a central role in extending the existence results to higher dimensions.

It is also important to specify in a more satisfactory manner the determinants of the order of entry into a particular market (e.g. see Hay, 1976). Frequently, the decision on whether or not to enter is being made by a number of firms at the same time. This raises entirely new issues. Will firms agree on an order of entry? If not, will there be a sudden rush to enter, with several firms entering simultaneously? Conversely, may the

prospective actions of future competitors deter entry altogether, and so prevent the development of potentially profitable markets? The issues are suggestive of a possible asymmetric war of attrition (Nalebuff and Riley, 1985).

Applications

For policy design, we must compare the socially optimal pattern of product location with the non-cooperative oligopoly solution. It may then be possible to separately understand market departures from optimality in (a) the design of products, (b) pricing strategies, and (c) the number of competing products.

Finally, a potentially important application of our model is to the theory of the product cycle in international trade theory. It is frequently contended that the U.S. acts as a leader in innovation but ultimately has its markets taken away by lower cost foreign producers. There is then the question of which kinds of products and innovations will be least susceptible to this kind of potentially destructive low cost competition. The answer has implications for R&D strategies both at the corporate and national levels as well as for the design of tariffs as part of an overall industrial policy. Our model appears well suited to studying product cycle dynamics since the central elements are the known order of entry into the market and the underlying asymmetry of costs.

APPENDIX

Proposition 4.1: For consumer preferences satisfying A1, any joint solution to the first-order conditions is a Bertrand–Nash equilibrium.

Proof

Consider first a joint solution, (P_x^*, P_y^*), with $F(\alpha_1^*) \in (0, 1)$. If Firm X (symmetrically, Firm Y) has a price better than P_x^* against P_y^*, this implies that there is some other price, P_x, with $F(\alpha_1^*) \in (0, 1)$, which is a local minimum of the profit function. But (1a)–(5a) prove this is impossible as any solution to the first-order condition with $F(\alpha_1^*) \in (0, 1)$ is always a local maximum.

$$\left.\frac{d^2\Pi_x}{dP_x^2}\right|_{(d\Pi_x/dP_x)=0} \sim -f(\alpha_1^*)C_x/P_x - f(\alpha_1^*) - f'(\alpha_1^*)(1 - C_x/P_x)/K \tag{1a}$$

$$\sim -2f(\alpha_1^*)^2 - f(\alpha_1^*)[1 - F(\alpha_1^*)]K - f'(\alpha_1^*)[1 - F(\alpha_1^*)] \tag{2a}$$

$$\leq -2f(\alpha_1^*)^2 - f'(\alpha_1^*)[1 - F(\alpha_1^*)] \tag{3a}$$

$$\leq -(\tfrac{3}{2})f(\alpha_1^*)^2 \tag{4a}$$

$$< 0. \tag{5a}$$

Equation (2a) follows by substitution of the first-order condition solution for C_x/P_x into (1a). Equation (3a) follows as $K > 0$. Equation (4a) follows as $-f'(\alpha_1^*)[1 - F(\alpha_1^*)] \leq (\frac{1}{2})f(\alpha_1^*)^2$ for any concave function F. To confirm this, note that the worst case occurs when f' is a negative constant. Then for any α_1, $1 - F(\alpha_1) \leq -\frac{1}{2}f(\alpha_1)^2/f'(\alpha_1)$. Equation (5a) follows by A1 as $f(\alpha_1^*) > 0$ for $F(\alpha_1^*) \in (0, 1)$.

The only other possibility is that at the joint solution, (P_x^*, P_y^*), one firm, say Firm X, receives a zero market share. In this case, by equation (7), $F(\alpha_1^*) = 1$ implies $f(\alpha_1^*) > 0$. From equation (5), this then requires that $P_x = C_x$. This is a best response for Firm X. Any lower price leads to losses, while higher prices maintain a zero market share. Since Firm Y has a positive market share (and $f(\alpha_1^*) > 0$) the previous argument proves the optimality of P_y^*.[8] □

Lemma A1 to *Proposition* 4.2: Under A1, $\psi'(\alpha_1^*) > 0$ for $\alpha_1^* \in (a, b)$.

Proof

Taking the derivative of equation (9) shows

$$\psi'(\alpha_1^*) = K\left[1 + \frac{2f^2 + f + f'[1-2F]}{(f + K[1-F])(f + KF)}\right] \tag{6a}$$

$$\geq K\left[1 + \frac{(3/2)f^2 + f}{(f + K[1-F])(f + KF)}\right] \tag{7a}$$

$$> 0 \quad \text{for} \quad \alpha_1^* \in (a, b). \tag{8a}$$

In this derivation, we have shortened $f(\alpha_1^*)$ to f and similarly for F and f'. Equation (7a) follows from (6a) by the concavity assumption A1. Again a triangular distribution is the worst case; with f' a constant, either positive or negative, $f'[1-2F] \geq -(\frac{1}{2})f^2$.

Substituting this inequality in (6a) yields the desired result. □

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[8] For his model, Lane (1980) constructs a unique joint solution to the first-order conditions. However, he does not consider whether the resulting prices form mutually best responses.

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