Journal of Public Economics 25 (1984) 1–12. North-Holland

DRAGON-SLAYING AND BALLROOM DANCING: THE PRIVATE SUPPLY OF A PUBLIC GOOD

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Received January 1983, revised version received March 1984

Many public goods typically are supplied by the efforts of a single individual. A purely self-interested agent could provide a public good if his own participation in the benefits justifies his cost. In this paper we model the decision of how a private individual decides when to take the initiative and pay for the provision of a public good. As an application of the optimal auctions literature to the public goods problem, the emphasis is placed on the effect of additional agents on potential supply. The free rider problem is shown to be less important as the population size of potential volunteers increases; we demonstrate conditions in which the first best is attained in the limit as the population size approaches infinity.

1. Introduction

Imagine that there exists a public good which has a positive value to everyone. Someone has to be the dragon-slayer and take the initiative to supply it. Common examples are donating a library, opening a window in a hot room, and jumping in to save a drowning swimmer. Of course while everyone will benefit from provision, only the person who takes the initiative has to pay the cost. We set aside the question of joint collective action to focus on the case of individual action.

If we agree in advance that the lowest cost individual should supply the good, then simply asking people to reveal their costs will not elicit truthful responses; there is a great advantage in being a free rider. One approach that is often tried could be called the bluffing equilibrium. The community presents one person with an ultimatum: 'Either you supply the good or we will just have to do without.'[1] If the bluff works, the good will be supplied forthwith, but otherwise perhaps never.

*It is a pleasure to thank Terence Gorman, Chris Harris, Eric Lander, Eric Maskin, Seth Masters, Dan Raff, John Riley, Kevin Roberts, Joel Sobel and two perceptive referees. Richard Zeckhauser provided the dancing instruction.

[1]Without some commitment mechanism, this ultimatum may not be credible, i.e. depending on how beliefs are specified for events outside the equilibrium this solution may not be subgame perfect; furthermore, mathematical biologists show that it is not evolutionarily stable [see Bishop et al. (1978)].

The approach we find most interesting is quite different. We examine the symmetric Nash equilibrium. Here, how long an individual waits before making his move is directly related both to his own cost and to his perception of the distribution of the costs of others. If the good has not been provided by the end of the delay appropriate to some individual, that individual is then justified in inferring that he has the lowest cost. Most importantly, at that time he will find his personal interests in exact accord with those of society and therefore he will decide to supply the public good.

In the field of theoretical biology, this type of waiting game, called a 'war of attrition', is studied by Bishop et al. (1978), Riley (1980), and Nalebuff and Riley (1984). These authors analyse how animals resolve fights over territorial claims and food supplies. While all parties are injured from the fighting, the side that is willing to continue longest gets the rewards. The winner's cost of fighting is determined by when his last opponent gives in.

In other models, it is the early bird that wins the fight. Holt and Sherman (1982) consider the use of queues as an allocation device. There is a known time when a ticket counter will open and the tickets will then be given out on a first-come, first-served basis. An individual's willingness to wait is demonstrated by his arrival time in the queue. Although the winners can be predicted in advance, they still have to wait; thus, the winners' costs depend on their own arrival times and are independent of when their challengers show up.[2] The problem of finding a public good provider is closer to a war of attrition because everyone's cost of waiting depends only on how long it takes to find a volunteer.

Like fighting and queueing, waiting is an effective revelation mechanism only when it hurts. Picturesque examples are that of holding out for a melting ice cube, suffering the pregnant pause before the first couple starts the ballroom dance, or sacrificing virgins to keep a dragon in abeyance. In the more realistic examples given earlier, people suffer until the library is built, perspire and are otherwise uncomfortable whilst waiting for the window to be opened, or risk allowing the swimmer to drown. How much of the value of the public good one is willing to forgo by waiting is directly related to one's cost of providing the good.[3] Thus, people have the

[2]Comparing our analysis to Holt and Sherman's, one referee discussed the informational differences: 'In the Holt and Sherman paper individuals arrive at the queue and obtain information; the queue may be larger or smaller than anticipated. If the expectation was that it would be large and it is in fact small, the individual may choose to leave the queue and return later. Holt–Sherman's analysis appears to be incorrect, in that they do not consider this possibility.' There is only one exception to our referee's point. If winning requires being first in line (e.g. there is only one ticket), then the line cannot be shorter than expected; there is no incentive to leave and return later. Our model of waiting for a single volunteer is informationally similar to the process of allocating a single ticket. Modelling the provision of a public good that requires several volunteers is significantly more complicated because, as suggested above, all the decisions will be interdependent.

[3]It also depends on one's cost of waiting relative to others. Here, everyone is assumed to be equally patient; Nalebuff (1982) discusses a model in which the cost of waiting differs across the population.

opportunity to prove how large their costs are by waiting. Eventually, the person with the lowest cost will become impatient and, having been outwaited, will decide to supply the public good.

A dramatic example of this is the game of 'Chicken.' Two cars race towards each other and the first driver to veer off the collision course is the chicken. The longer each driver waits, the smaller is the value from turning as collision is more likely to be inevitable. The driver who gives in does so only after his opponent has proved to him (by a willingness to wait longer) that his costs (in terms of pride in this example) are larger. When it comes to proving one's costs, inaction speaks louder than words.

This approach to public good provision can be viewed as an application of the optimal auctions literature. Because our focus is on when public goods will be provided, we emphasize the effects from changes in population size and from changes in the distribution of the population. To demonstrate this approach, we present a model in which there exists a unique symmetric solution with the following properties:

Theorem 1. Each agent's optimal waiting time increases monotonically with respect to his cost of providing the good and is directly proportional to $N-1$ when there are N agents.

Theorem 2. Increasing the number of participants, although increasing everyone's waiting time, also raises their expected utility.

Theorem 3. An agent's gain in expected utility from an additional player rises with his cost of providing the good.

Theorem 4. The good will always be supplied by the agent with the minimum cost. When the distribution of the minimum cost agent becomes 'riskier', this change is expected to prolong the waiting time before the good is supplied.

Theorem 5. Adding another participant increases the probability that the good will be supplied before an exceedingly long time, but may either increase or decrease the chance of someone acting almost immediately.

Theorem 6. Under certain conditions, in the limit as the population size approaches infinity, the free rider problem vanishes; the expected waiting time is zero and the first-best result is attained.

When there is a larger number of potential volunteers, the free rider problem becomes less important even though everyone waits longer; below, we demonstrate conditions under which, in the limit as the population size approaches infinity, the public good will be provided immediately. Increasing the number of agents improves each agent's expected utility as they each are now less likely to have to provide the public good and because the good may be provided sooner. What is perhaps surprising is that the person who benefits the most is the one who must wait for others to supply the good. A person with the highest cost will never supply the good, yet he has the greatest gain from additional agents.

2. The model

The value of the public good at present is 1 for all agents and the value in the future is discounted exponentially. An agent's cost of supplying the public good is c which is distributed in the population with a cumulative density (or distribution) function, $F(c)$, which is differentiable and $F'(c)=f(c)$. Because no one with a cost $c>1$ would ever supply the good, our attention can be restricted without loss of generality to distributions of c that are supported on $[0,1]$. Finally, each agent assumes (correctly) that there are n other agents whose costs are independently and identically distributed with density $f(c)$.

We use the revelation principle to show that a monotonic function $T(n+1,c)$ exists which is the optimal waiting time for an agent with cost c when there are n other agents.[4] Consider one agent's choice when everyone else is acting according to $T(n+1,c)$. An agent with true cost c can of course pretend to have a higher (or lower) cost c^*. Since the waiting time monotonically increases with cost, choosing a waiting time is equivalent to choosing a c^* and acting according to $T(n+1,c^*)$.[5] If an agent pretends to have a cost c^*, his expected utility will be his true net benefit from supplying the public good $(1-c)$ discounted to time $T(n+1,c^*)$ if c^* is indeed the lowest cost, plus his expected discounted value if the good is supplied by a lower (than c^*) cost player:

$$\mathrm{E}[U(c,c^*)]=(1-c)\,\mathrm{e}^{-T(n+1,c^*)}[1-F(c^*)]^n + \int_0^{c^*} \mathrm{e}^{-T(n+1,x)} n f(x)[1-F(x)]^{n-1}\,\mathrm{d}x. \quad (1)$$

[4]This is related to the point first developed by Mirrlees (1976) in his synthesis of optimal tax theory. For any solution to the principal–agent problem, there is an equivalent solution in which it is optimal for the agents to reveal the truth about their unobservable parameters. Given any monotonic waiting time function, $T(n+1,x)$, agents will choose an optimal strategy, $c^*(c)$, and act according to $T(n+1,c^*(c))$. If agents were initially presented with $T(n+1,c^*(x))$, then they would choose x equal to their true cost. This is the constraint used to define the optimal waiting time function, $T(n+1,c)$.

[5]Agents are really choosing an optimal reservation level of expected losses, e^{-T}. If the cost from waiting was not exponential, then the optimal waiting time would be different but the expected waiting cost would be exactly the same. In the auctions literature this result, known as the 'revenue equivalence theorem' [Milgrom and Weber (1980) and Riley and Samuelson (1981)], is that with risk neutral agents all bidding schemes must yield the same expected revenue.

As an agent waits, information is continually being revealed; if no one has volunteered by $T(n+1,c^*)$, then each agent knows that everyone else's costs are above c^*. However, the whole process is predictable. Agents know that their decision to act is relevant only if no one else has previously volunteered; they can predict in advance what they will have learned at their time to act. Therefore, eq. (1) can be re-evaluated at any time period before $T(n+1,c^*)$ and the decision to act at $T(n+1,c^*)$ remains optimal when c^* satisfies the first-order condition:

$$\mathrm{e}^{-T(n+1,c^*)}[1-F(c^*)]^{n-1}\{-(1-c)[1-F(c^*)]\,\partial T/\partial c^* \\ -n(1-c)f(c^*)+nf(c^*)\}=0. \tag{2}$$

This implies:

$$\partial T(n+1,c)/\partial c|_{c*}=ncf(c^*)/[(1-c)(1-F(c^*))], \tag{3}$$

which defines $c^*(c)$ for any given function $T(n+1,c)$. When an agent chooses c^* optimally, we know from the definition of T that he must find it in his best interests to choose $c^*=c$. Thus, the first-order condition requires

$$\left.\frac{\partial \mathrm{E}[U(c,c^*)]}{\partial c^*}\right|_{c^*=c}=0 \Rightarrow \frac{\partial T(n+1,c)}{\partial c}=\frac{ncf(c)}{[(1-c)(1-F(c))]}. \tag{4}$$

Eqs. (2) and (4) imply that the second-order conditions will be satisfied, since

$$\operatorname{sign}(\partial \mathrm{E}U/\partial c^*)=\operatorname{sign}(c-c^*). \tag{5}$$

To solve for T we must integrate eq. (4). Fortunately, a person with zero cost will always find it optimal to act immediately. Hence, we have the initial condition

$$T(n+1,0)=0, \quad \forall n. \tag{6}$$

We have implicitly shown that a function $T(n+1,c)$ does exist:[6] it is the integral of eq. (4) subject to eq. (6). Agents assuming this function will generate it. The fact that T_c is unambiguously positive proves that waiting

[6]Technically, the proof has only shown that there exists a unique monotonic differentiable function mapping costs on to waiting times. It is relatively easy to prove that any solution must be monotonic and differentiable [for example, see proofs in Nalebuff (1982), Milgrom and Weber (1981), and Maskin and Riley (1981)]. Monotonicity follows from the fact that agents with higher costs also have a greater marginal utility from waiting. Because the distribution of waiting times must be both gapless and atomless, continuity and monotonicity leads to differentiability almost everywhere.

time increases directly with cost. Because the derivative is exactly proportional to n and the starting condition is always zero we have

$$T(n+1,c)=nT(2,c). \tag{7}$$

This completes the proof of our first result. For any given cost, c, each agent's optimal waiting time is proportional to n when there are $n+1$ agents with the same cost distributions.

To see how expected utility changes with respect to n, first start at a point where the answer is easy, $c=0$. A person with zero cost will always act immediately and thus his expected utility is constant at 1 independent of the total number of agents:

$$\partial \mathrm{E}[U(0,0]/\partial n=0, \quad \forall n. \tag{8}$$

Agents with higher costs receive relatively more benefits as n increases:

$$\partial^2 \mathrm{E}[U(c,c)]/\partial n\, \partial c=\partial^2 \mathrm{E}[U(c,c)]/\partial c\, \partial n \tag{9}$$

$$=\partial[-\mathrm{e}^{-nT(2,c)}[1-F(c)]^n]/\partial n \tag{9i}$$

$$\geqq 0. \tag{9ii}$$

The proof is simplified by recalling from eq. (2) that c^* is chosen optimally and is not a function of n. Thus, we need be concerned only with the partial derivative. A person with higher costs benefits more from additional agents because he values more highly both the delay in his provision of the good and the increased probability that he will not indeed be the lowest cost supplier.

The argument above proves that an increase in the number of agents is a Pareto improvement in the sense that each type of agent has a higher welfare. The fact that each person waits longer before acting himself [eq. (7)] is more than offset by the presence of an additional agent; the good will be provided at a lower expected cost and everyone has a greater chance of being a free rider. In particular, the agent who will never supply the good ($c=1$) and thus depends on being a free rider will benefit the most; from his perspective, the addition of an agent changes the distribution of the minimum cost in a manner sufficiently favorable to offset the lengthened waiting time of the agent who will act.

Although everyone is better off, this does not directly imply that the good will be provided sooner. A natural question to ask is whether the presence of an additional agent lowers the expected discount factor. For small groups, the answer is ambiguous. However, as the population size becomes sufficiently large; the expected discount factor converges. Below, we demonstrate conditions under which in the limit as the population size approaches infinity, the free rider problem disappears and the first-best result is attainable.

As the population size approaches infinity, the utility loss from waiting converges to $\hat{c}$, where $\hat{c}$ is the minimum cost with positive density:

$$\hat{c}=\sup\{x: F(x)=0\}; \tag{10}$$

if $\hat{c}=0$, then the free rider problem vanishes as the expected cost of waiting approaches zero.

The expected discount factor can be bounded above and below using integration by parts:

$$\mathrm{E}[\mathrm{e}^{-T_{\min}}]=\int_0^1 (n+1)\,\mathrm{e}^{-T(n+1,x)}[1-F(x)]^n f(x)\,\mathrm{d}x \tag{11}$$

$$=1-\int_0^1 \frac{nx}{1-x}\mathrm{e}^{-T(n+1,x)}[1-F(x)]^n f(x)\,\mathrm{d}x. \tag{11i}$$

Since $\hat{c}/(1-\hat{c})<x/(1-x)$ over the non-zero range of the integral, substitution into (11i) provides an upper bound for $\mathrm{E}[\mathrm{e}^{-T_{\min}}]$:

$$(n+1)\int_0^1 \mathrm{e}^{-T(n+1,x)}[1-F(x)]^n f(x)\,\mathrm{d}x$$

$$\leqq 1-\frac{n\hat{c}}{1-\hat{c}}\int_0^1 \mathrm{e}^{-T(n+1,x)}[1-F(x)]^n f(x)\mathrm{d}x \tag{12}$$

$$\leqq\frac{(n+1)(1-\hat{c})}{n+1-\hat{c}}. \tag{12i}$$

Alternatively, if we only integrate over the interval $[\hat{c},\hat{c}+\varepsilon]$, then

$$\frac{x}{1-x}\leqq\frac{\hat{c}+\varepsilon}{1-\hat{c}-\varepsilon}$$

and this provides a lower bound for $\mathrm{E}[\mathrm{e}^{-T_{\min}}]$:

$$(n+1)\int_0^1 \mathrm{e}^{-T(n+1,x)}[1-F(x)]^n f(x)\,\mathrm{d}x$$

$$\geqq(n+1)\int_{\hat{c}}^{\hat{c}+\varepsilon} \mathrm{e}^{-T(n+1,x)}[1-F(x)]^n f(x)\mathrm{d}x \tag{13}$$

$$\geqq 1-\mathrm{e}^{-nT(2,\hat{c}+\varepsilon)}(1-F(\hat{c}+\varepsilon))^n - \frac{n(\hat{c}+\varepsilon)}{1-\hat{c}-\varepsilon}$$

$$\times \int_{\hat{c}}^{\hat{c}+\varepsilon} \mathrm{e}^{-T(n+1,x)}[1-F(x)]^n f(x)\mathrm{d}x \tag{13i}$$

$$\geqq [1-\mathrm{e}^{-nT(2,\hat{c}+\varepsilon)}(1-F(\hat{c}+\varepsilon))^n]\frac{(n+1)(1-\hat{c}-\varepsilon)}{n+1-\hat{c}-\varepsilon}. \tag{13ii}$$

Taking the limits of (12i) and (13ii) as n approaches infinity shows

$$1-\hat{c}-\varepsilon \leqq \lim_{n\to\infty} \mathrm{E}[\mathrm{e}^{-T_{\min}}] \leqq (1-\hat{c}). \tag{14}$$

As this remains true for any positive ε, the limit of $\mathrm{E}[\mathrm{e}^{-T_{\min}}]$ must be $1-\hat{c}$.

A similar argument demonstrates that the expected waiting time converges[7] to $\hat{c}/(1-\hat{c})$. Intuitively, for large n, $C_{\min}$ is close to $\hat{c}+1/[f(\hat{c})(n+1)]$ and $T(C_{\min})$ is close to $T(\hat{c})+T'(\hat{c})(C_{\min}-\hat{c})$:

$$T(C_{\min}) \approx 0+\frac{n\hat{c}f(\hat{c})}{(1-\hat{c})[1-F(\hat{c})]}\cdot\frac{1}{[f(\hat{c})(n+1)]} \approx \frac{\hat{c}}{1-\hat{c}}. \tag{15}$$

Thus, if $\hat{c}$ is zero, then the expected waiting time also approaches zero.

Calculating changes in the expected waiting time when the population size is finite is more difficult. We are only able to show how an increase in the number of potential volunteers changes the distribution of waiting times at both tails: there is always a greater chance that the good will be supplied before an extremely long time; whether there is a greater chance of almost immediate action depends on $\hat{c}$.

When there are $n+1$ agents, the probability that the good is supplied before time T is

$$P(T)=1-[1-F(\bar{c})]^{n+1}, \quad \text{where } \bar{c} \text{ is defined by} \quad nT(2,\bar{c})=T. \tag{16}$$

We are interested in how this probability changes with an increase in n. The direct effect of an additional agent always raises the chance of action. However, the cost, $\bar{c}$, corresponding to time T is also lowered, which reduces the fraction of potential volunteers, $F(\bar{c})$. While the sign of $\mathrm{d}P(T)/\mathrm{d}n$ is ambiguous, we show in the appendix that at very large values of T, $\mathrm{d}(P(T))/\mathrm{d}n$ is always positive, while at very small values of T, $\mathrm{d}(P(T))/\mathrm{d}n$ is positive if and only if $\hat{c}$ is zero. This follows our intuition because when $\hat{c}=0$ the expected waiting time converges to zero, while if $\hat{c}>0$, then the expected waiting time $(\hat{c}/1-\hat{c})$ remains greater than zero.

[7]To guarantee convergence requires $F(1)=1$; otherwise there is always a finite probability of an infinite waiting time.

For any population size, the expected waiting time before the good is supplied is increased when the distribution of the minimum cost becomes 'riskier' [in the sense defined by Rothschild and Stiglitz (1970)]. We consider changes in the distribution of F that hold the expected *minimum* of the $n+1$ c_i's constant because the first volunteer will be the player with the lowest c. A similar argument shows that when we consider mean preserving spreads of $F(x)$, then all players are expected to act later.[8]

The expected waiting time depends both on the distribution of costs and the waiting rule, $T(n+1,x)$; a mean preserving increase in risk changes both. Let $F(c,z)$ represent the distribution function for c with z as the risk shift parameter. Formally, increasing z corresponds to a mean preserving increase in risk if, for some $\tilde{c}$:

$$F_z(c,z) \geqq 0, \quad \text{for} \quad c \leqq \tilde{c},$$

$$F_z(c,z) \leqq 0, \quad \text{for} \quad c > \tilde{c},$$

and

$$\int_0^1 [1-F(x,z)]^n F_z(x,z)\,\mathrm{d}x = 0. \tag{17}$$

Repeated integration by parts shows that the expected waiting time is

$$\mathrm{E}[T_{\min}(z)] = (n+1)\int_0^1 T(n+1,x)[1-F(x,z)]^n f(x,z)\,\mathrm{d}x \tag{18}$$

$$= \int_0^1 \frac{nx}{(1-x)}[1-F(x,z)]^n f(x,z)\,\mathrm{d}x \tag{18i}$$

$$= \frac{n}{n+1}\int_0^1 \frac{[1-F(x,z)]^{n+1}}{(1-x)^2}\,\mathrm{d}x. \tag{18ii}$$

As the risk parameter increases, the expected waiting time rises:

$$\frac{\mathrm{dE}[T_{\min}(z)]}{\mathrm{d}z} = -n\int_0^1 \frac{[1-F(x,z)]^n}{(1-x)^2} F_z(x,z)\,\mathrm{d}x \tag{19}$$

$$\geqq \frac{-n}{(1-\tilde{c})^2}\int_0^1 [1-F(x,z)]^n F_z(x,z)\,\mathrm{d}x \tag{19i}$$

$$\geqq 0. \tag{19ii}$$

[8]The proofs are very similar. The first part of the proof uses l'Hôpital's rule. For this argument to be valid in the case of mean preserving spreads of $F(x)$ requires an additional assumption; either the maximum x_i must be strictly less than one or we must restrict our attention to mean preserving risk increases in F such that $f(1)$ is constant.

The inequality in (19i) is derived from the relatively greater weights placed on the positive range of F_z than on the negative portion. The definition of a mean preserving spread in eq. (17) leads to the completion of the proof in (19ii). Hence, the process of soliciting volunteers can be speeded up when it is possible on average to move people away from extreme cost positions.

3. Conflict resolution

The way in which people decide who should supply a public good comes down to 'brinkmanship'. Thomas Schelling (1960), in his study of conflict, described for adults what children understand perfectly: 'Brinkmanship involves getting on to the slope where one may fall in spite of one's own best efforts to save himself, dragging his adversary with him.' The idea behind brinkmanship is to make a threat that leaves something to chance. The discount loss in waiting for the public good to be supplied is equivalent to the loss from the probability of an accident in going onto the edge of a slope.[9] In one of our first examples, those on the beach waiting for someone else to jump in and save the drowning swimmer lose utility because there is an increasing probability that it is too late to prevent the swimmer from drowning.

4. Summing up

There are more gracious ways to find a volunteer than relying on brinkmanship. Binding agreements combined with side payments can always produce a superior outcome. But the world does not always provide an authority that can enforce the agreements to make them binding. Social customs against bribery often raise prohibitive barriers against making side payments. Relying on a private individual to supply a public good is usually the last option. When the distribution of costs is more extreme, the free rider problem is more severe as society waits longer hoping that there is a low cost individual preparing to volunteer. Fortunately, as the population size increases, the free rider problem is less important. In the limit as the population size approaches infinity, the utility cost of waiting converges to the minimum of the cost distribution; when $\hat{c}=0$, the first-best is attained. This suggests that the free rider problem will be greatest when there is only a

[9]Imagine a rowboat heading towards a waterfall with two passengers neither of whom wants to row. A person who makes the threat 'I won't row no matter what' is likely to reconsider when the other person makes the same threat. However, it is rational for a person to promise not to start rowing until the probability of unavoidable disaster is P. The probability will be directly related to his cost of rowing, just as the time delay before a person provides a public good is related to his cost. Even if the rowers could announce their reservation probabilities to each other, the person with the lower cost would still have to wait until his reservation probability materialized as otherwise both players would have an overriding incentive to bluff. See Nalebuff (1982) for a lengthier discussion of this issue.

limited pool of potential volunteers. Even in small groups, company is appreciated while waiting for a volunteer; more people in the same boat helps make everyone better off.

Appendix

This appendix provides the proof of the proposition that increasing the number of agents always results in a greater chance that the good will be supplied before an extremely long time and also a greater chance of almost immediate action if and only if $\hat{c}=0$.

Starting with the definition of $P(T)$ and $\bar{c}$ in eq. (16), we differentiate with respect to n:

$$\frac{\mathrm{d}P(T)}{\mathrm{d}n}=[1-F(\bar{c})]^{n+1}\left\{-\log[1-F(\bar{c})]+\frac{(n+1)f(\bar{c})}{[1-F(\bar{c})]}\frac{\mathrm{d}\bar{c}}{\mathrm{d}n}\right\}. \tag{A.1}$$

where

$$\mathrm{d}\bar{c}/\mathrm{d}n=-[T(2,\bar{c})(1-\bar{c})(1-F(\bar{c}))]/[\bar{c}f(\bar{c})n]. \tag{A.2}$$

Substitution of (A.2) into (A.1) yields:

$$\frac{\mathrm{d}P(T)}{\mathrm{d}n}=-[1-F(\bar{c})]^{n+1}\left\{\log[1-F(\bar{c})]+\frac{(n+1)(1-\bar{c})}{[n\bar{c}]}T(2,\bar{c})\right\}. \tag{A.3}$$

In the proofs, because T and $\bar{c}$ are monotonically related, the limit of $\mathrm{d}P/\mathrm{d}n$ as T approaches infinity is the same as the limiting value as $\bar{c}$ approaches one. Similarly, to calculate the limiting value of $\mathrm{d}P/\mathrm{d}n$ at T equals zero, it is equivalent to look at the limit as $\bar{c}$ approaches its minimum value, $\hat{c}$. When $F(\hat{c})=0$ and $F(1)=1$, then $P(0)=0$ and $P(\infty)=1$ independent of n; $\mathrm{d}P/\mathrm{d}n$ must be zero at both endpoints.[10] Thus, the argument depends on proving that the derivative approaches zero from above in both cases.

As a first step, repeated application of l'Hôpital's rule proves:

$$\lim_{\bar{c}\to 1}(1-\bar{c})T(2,\bar{c})=\lim_{\bar{c}\to 1}\frac{\bar{c}f(\bar{c})/[(1-\bar{c})(1-F(\bar{c}))]}{1/[1-\bar{c}]^2} \tag{A.4}$$

$$=\lim_{\bar{c}\to 1}\frac{(1-\bar{c})f(\bar{c})}{[1-F(\bar{c})]}=\lim_{\bar{c}\to 1}\frac{-f(\bar{c})}{-f(\bar{c})} \tag{A.4i}$$

$$=1. \tag{A.4ii}$$

[10]If $F(1)<1$, then straightforward substitution combined with l'Hôpital's rule shows that $\mathrm{d}P(\infty)/\mathrm{d}n>0$. Similarly, if $F(\hat{c})>0$, then direct substitution in eq. (A.3) shows that $\mathrm{d}P(0)/\mathrm{d}n\geq 0$.

To compare the terms in eq. (A3) in the limit as $\bar{c}$ nears 1, observe that $\log[1-F(\bar{c})$ becomes arbitrarily large while $-(1-\bar{c})T(2,c)$ is bounded. Hence, at some sufficiently large value of T, the positive term dominates and the derivative, $\mathrm{d}P/\mathrm{d}N$, tends toward zero from above; eventually, larger groups are more likely to supply the good.

At time zero, the increased chance of a volunteer, $\mathrm{d}P(T)/\mathrm{d}n$, starts at zero. We now show when this probability initially rises with T. If over a short enough time period $\mathrm{d}P/\mathrm{d}n$ is positive, larger groups will also have a greater chance of supplying the good soon. To calculate the cross derivative again requires repeated application of l'Hôpital's rule:

$$\lim_{\bar{c}\to\hat{c}} \frac{\mathrm{d}^2 P(T)}{\mathrm{d}n\,\mathrm{d}\bar{c}} = \lim_{\bar{c}\to\hat{c}} \left\{ \frac{f(\bar{c})}{1-F(\bar{c})} - \frac{n+1}{n}\left[\frac{f(\bar{c})}{1-F(\bar{c})} - \frac{T(2,\bar{c})}{\bar{c}^2}\right]\right\}. \tag{A.5}$$

How we proceed depends on whether $\hat{c}=0$. When $\hat{c}=0$, we again use l'Hôpital's rule to show:

$$\lim_{\bar{c}\to 0} \frac{\mathrm{d}^2 P(T)}{\mathrm{d}n\,\mathrm{d}\bar{c}} = \lim_{\bar{c}\to 0} f(\bar{c}) \left\{ 1 - \frac{n+1}{n}\left[1 - \frac{\bar{c}}{2\bar{c}(1-\bar{c})}\right]\right\} = f(\hat{c})\frac{n-1+2\hat{c}}{2n(1-\hat{c})} \tag{A.6}$$

$$\geqq 0, \quad \text{as} \quad n \geqq 2. \tag{A.6i}$$

However, if $\hat{c}>0$, then the last term in eq. (A5) drops out and the limit is negative. For $\hat{c}>0$:

$$\lim_{\bar{c}\to\hat{c}} \frac{\mathrm{d}^2 P(T)}{\mathrm{d}n\,\mathrm{d}\bar{c}} = \frac{-f(\hat{c})}{n} < 0. \tag{A.7}$$

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