THE ACCOUNTING REVIEW
Vol. 66, No. 4
October 1991
pp. 669–693

Efficiency of Asset Valuation Rules Under Price Movement and Measurement Errors

Suk S. Lim
University of Alberta
Shyam Sunder
Carnegie Mellon University

SYNOPSIS AND INTRODUCTION: Errors arise in measuring changes in prices of assets due to imperfection and incompleteness of asset markets. Furthermore, the rates of price-change, and the magnitudes of errors of measurement vary and are often correlated across assets. Suppose we characterize an economy by means and variances of price changes for individual goods and of measurement errors in these changes as well as by the degree of diversification in the asset portfolios held by individual firms. In such an economy, the linear valuation rule that yields the most efficient estimate of change in the economic value of these asset portfolios is the one that minimizes the mean squared error (*MSE*). This paper presents a linear aggregation model of valuation to help understand how the minimum *MSE* valuation rule is affected by various parameters that characterize the economy, and the circumstances under which historical-cost valuation rule yields a (statistically) more precise estimate of the unobserved economic value of firms' assets than the current valuation rule. The analytical findings of the paper are consistent with the reluctance of accountants to depart from historical cost in spite of the existence of low inflation, and in spite of scholarly critiques of this valuation rule by Chambers (1966), Edwards and Bell (1961), Sterling (1970) and others. They are also consistent with the use of specific price indexes by most firms to prepare *SFAS 33* disclosures. Several testable implicatons of the results are provided.

We have benefited from comments received on an earlier draft of this paper from participants in accounting workshops at University of Minnesota, University of Alberta, McMaster University, University of Arizona, Arizona State University, and University of Florida. Comments of Amin Amershi, Jack Hughes, and anonymous referees of *The Accounting Review* were valuable in revising the paper. Financial support for this research, provided by McKnight Foundation, Honeywell Foundation, Arthur Andersen, Royal Bank of Canada, and National Science Foundation under Contract SES89-12552, is gratefully acknowledged.

Submitted October 1987 and July 1990.
Accepted May 1991.

A direct comparison of the characteristics of valuation rules is complicated by the heterogeneity of the decision contexts in which accounting numbers are used. We use the mean squared error (MSE) between the principal value and its various estimators to rank the latter. Using this criterion, previous simpler models that ignore the presence of measurement errors in price changes have shown that the use of increasingly detailed price indexes yields more precise valuation; current valuation is the most precise valuation rule because it uses the most detailed set of indexes (Sunder 1978). We show that this basic result does not hold when the measurement of price changes is subject to errors. As the magnitude of these measurement errors increases relative to the magnitude of price changes, the most accurate valuation rule requires a less detailed set of price indexes. A key implication of this result is that the existence of inflation or deflation is not sufficient for general-price-level valuation, specific-price-index valuation, or current valuation to dominate historical-cost valuation as an estimator of the economic value of firms' assets. Historical-cost valuation is dominated by others only when the magnitude of price changes are large relative to the errors of measurement in price changes.

Key Words: *Valuation rules, Valuation errors, Price indexes, Current valuation, Inflation accounting.*

THE remainder of this article is organized in four sections. Section I of the paper summarizes a model of valuation rules as linear aggregations and some key results from prior work that assumes the absence of measurement errors. In section II, efficient sets of estimators in the presence of measurement errors are analyzed and it is shown that, under specific assumptions about parameters, the marginal reduction in *MSE* from increasingly specific price indexes keeps decreasing. In section III we derive conditions under which historical-cost valuation and some specific price-index valuation rule, are globally minimum *MSE* estimators. Section IV presents empirically testable implications and suggests directions for further research.

I. Valuation Rules as Linear Aggregations

By analyzing environments where price changes are measured *without* error, Sunder (1978) shows that the bias and *MSE*[1] associated with a valuation rule $R_{k\lambda}$ are given by:

$$Bias(R_{k\lambda})=E_{\underline{w}}E_{\underline{r}}(R_{k\lambda}-R_{n,1})=0, \tag{1}$$

[1] For statistical analysis of valuation rules as linear aggregates, see Ijiri (1967, 1968) and Lev (1969). For ranking valuation rules by mean squared error criterion see Hall (1982); Hall and Shriver (1990); Ijiri and Noel (1984); Shriver (1986, 1987); Sunder (1976, 1978); Sunder and Waymire (1983, 1984); Tippett (1987); and Tritschler (1969). When markets are incomplete, it is not usually possible to rank unambiguously the values of various asset portfolios from the view point of all agents (Beaver and Demski 1979). However, if the performances of these agents are defined over means and variances of returns from these baskets, different valuation rules can be ranked unanimously in spite of incomplete markets (Eckern and Wilson 1974; Radner

and

$$MSE(R_{k\lambda}) = E_{\underline{w}}E_{\underline{r}}(R_{k\lambda} - R_{n,1})^2$$

$$= \frac{1}{\rho}\underline{\omega}'(\underline{\sigma} + \underline{\dot{\mu}}) - \frac{1}{\rho}\sum_{u=1}^{k}\frac{\underline{\omega}_u'(\Sigma_{uu} + \underline{\mu}_u\underline{\mu}_u')}{\underline{\omega}_u'\underline{e}}\underline{\omega}_u. \qquad (2)$$

To explain the notation briefly (see Sunder 1978 for details), we observe that the model assumes n distinct goods in the economy. The relative abundance of these goods is specified by an unchanging vector $\underline{\omega}$ of relative weights.[2] Because the elements of $\underline{\omega}$ sum to 1, $\omega_i = 0.1$, implies that the value of good i in the economy amounts to 10 percent of the total value of all goods in the economy. These weights are used to construct price indexes from prices of individual goods or assets. The vector of relative weights for the asset portfolio of individual firms, $\underline{w}$, has a multinomial distribution with parameter $\underline{\omega}$.[3] Vector $\underline{r}$ denotes the relative price changes (inflation or deflation) for n goods during an arbitrary time interval 0 to 1; $\underline{\mu}$ and Σ are expectation and variance, respectively, of $\underline{r}$. Let $S(n,k)$ be the number of distinct ways in which a set of n goods can be partitioned into k subsets.[4] Taking weighted average (using weights $\underline{\omega}$) of price changes for goods included in each subset produces k price indexes. When a good is included in the asset portfolio of a firm as well as in a given index u, the firm adjusts the historical cost of the good on its books by this price index to arrive at its estimated current value. Percentage change in the sum of the estimated current value of all assets in the firm's portfolio, relative to its historical cost, is denoted by $R_{k\lambda}$ if λth of the $S(n,k)$ possible k-index valuation rules (ordered in some arbitrary manner) is used to estimate the current value. Note that when $k=n$ or $k=1$, $S(n,k)=1$, which means that there is only one possible way each of partitioning n goods into n and 1 partitions, respectively; use of the former yields current valuation rule denoted by $R_{n,1}$, while the use of the latter yields general price-level valuation (GPL) denoted by $R_{1,1}$. In addition, we could denote historical-

---

1974). Also, in an economy-wide sense, neither overvaluation nor undervaluation is desirable because both lead to a suboptimal resource allocation. Since mean square error metric deals with errors in both directions, it is not an unreasonable choice for a loss function.

[2] Unchanging $\underline{\omega}$ (and $\underline{w}$ to be defined below) imply that this valuation model takes the asset composition of firms and the economy as given, it does not try to explain the dynamics of change in asset composition and prices.

[3] In effect, it is assumed that the asset portfolio of each firm is constructed by making ρ independent draws with replacement from an urn in which each asset is represented by a different colored ball and the relative proportion of each color is given by $\underline{\omega}$. Thus, each firm is statistically identical in the sense that its asset basket is drawn from the same distribution. This structure ignores the dependencies that may exist across firms (e.g., among firms within an industry). For analysis of valuation rules in industry-segmented economies, see Lim and Sunder (1990). From properties of the multinomial distribution, $E(\underline{w}) = \underline{\omega}$ and $\mathrm{Cov}(w_i, w_j) = \omega_i(1-\omega_i)/\rho$ if $i=j$, and $-\omega_i\omega_j/\rho$ otherwise. Parameter ρ can be interpreted as a diversification parameter; as ρ increases, the vector of asset proportions for individual firms statistically converges to the economy-wide vector of asset proportions $\underline{\omega}$ (see Feller 1968 and Sunder 1978).

[4] This is Stirling Number of the Second Kind given by:

$$S(n,k) = \sum_{j=0}^{k-1}\frac{(-1)^j(k-j)^n}{\{j!(k-j)!\}}.$$

See Apostol (1967, 594).

valuation rule by $R_{0,1}$ because there is only one possible way of using no price indexes. For all other values of k, there are $S(n,k)>1$ ways of constructing k price indexes. $\underline{\sigma}$ is the vector of diagonal elements of Σ, and $\dot{\underline{\mu}}$ is the vector of squared elements of $\underline{\mu}$. $\underline{\omega}_u$ and $\underline{\mu}_u$ are vectors containing those elements of $\underline{\omega}$ and $\underline{\mu}$, respectively, that are included in the uth price index, and Σ_{uu} is the submatrix of Σ consisting of all those rows and columns that correspond to the goods contained in the uth index. Finally, $\underline{e}$ is a vector of unit elements with appropriate length. A numerical example of valuation using rules of this class is given in Lim and Sunder (1990, 171–2).

Sunder (1978) shows that bias (1) for all estimators of this class is zero.[5] Mean squared error (2) monotonically decreases in fineness[6] of the partition used in constructing $R_{k\lambda}$. It immediately follows that the *MSE* is largest for the coarsest estimator (GPL valuation, $R_{1,1}$) and is smallest for the finest estimator (current valuation $R_{n,1}$). The intuition behind this result is that the use of coarser price indexes in valuation causes a greater mismatch between the relative weights assigned to changes in price of individual goods, and the relative proportions in which each good may be held in the actual asset portfolios of individual firms. The *MSE*s of other linear valuation rules lie between these two extremes.

We can add historical-cost valuation to this family by defining it as a zero-index valuation, $R_{0,1}\equiv 0$, because historical valuation of an unchanging basket of assets does not change over time. The historical-cost valuation is the only member of the family with a nonzero bias. In absence of measurement errors, its *MSE* in equation (4) is strictly higher than the *MSE* of the other linear valuation rules (see Sunder 1978).

$$Bias(R_{0,1})=E_{\underline{w}}E_{\underline{r}}(R_{0,1}-R_{n,1})=-\underline{\omega}'\underline{\mu}, \tag{3}$$

and

$$MSE(R_{0,1})=E_{\underline{w}}E_{\underline{r}}(R_{0,1}-R_{n,1})^2=\frac{1}{\rho}\underline{\omega}'(\underline{\sigma}+\dot{\underline{\mu}})+\left(1-\frac{1}{\rho}\right)\underline{\omega}'(\Sigma+\underline{\mu}\underline{\mu}')\underline{\omega}. \tag{4}$$

The latter property of the valuation rules can readily be seen under label *MV* in the sixth column of table 1, which lists the *MSE* values, associated with all 16 possible valuation rules in a four-good economy for a numerical example with the following parameters:[7]

- relative weight of each good in the economy, $\underline{\omega}'=(0.2, 0.3, 0.4, 0.1)$;
- expected price change for each good in the economy, $\underline{\mu}'=(0.1, 0.2, 0.6, 0.4)$; and
- covariance matrix of relative price changes for goods in the economy,

[5] The total number of distinct estimators in this class for an n-good economy is given by:

$$\sum_{k=1}^{n} S(n,k)$$

where $S(n,k)$ is as given in footnote 4.

[6] Price index set x is finer than price index set y if and only if all the goods contained in each one of the price indexes of set x are also included in some price index of set y. For a five-good economy, for example, the index set {(1.2), (3), (4), (5)} is finer than {(1,2), (3,4), (5)} but is not comparable in fineness to {(1), (2,3), (4,5)}. Note that increasing k does not necessarily mean a finer valuation rule or a smaller mean squared error.

[7] Although this numerical example uses zero covariances, the model does not require them to be zero.

Table 1
Properties of Estimators of Current Value in a Four-Good Economy: Example 1

1	2	3	4	5	6	7
				Economy-Wide Average Errors		
k	$S(n,k)$	Serial Number	Partition	$MSE(\tilde{R}_{k\lambda})$ =	$MV(\tilde{R}_{k\lambda})$ +	$MR(\tilde{R}_{k\lambda})$
0	—	—	Historical	$1.260+2.214/\rho$*	$1.260+2.214/\rho$*	0*
1	1	0	GPL(abcd)	$0.66+2.214/\rho$*	$2.214/\rho$*	0.66*
2	7	1	(a,bcd)	$0.66+2.380/\rho$	$1.765/\rho$	$0.66+0.615/\rho$
		2	(b,acd)	$0.66+2.174/\rho$	$1.205/\rho$	$0.66+0.969/\rho$
		3	(c,abd)	$0.66+1.329/\rho$*	$0.756/\rho$*	$0.66+0.573/\rho$
		4	(d,abc)	$0.66+2.162/\rho$	$1.911/\rho$	$0.66+0.251/\rho$*
		5	(ab,cd)	$0.66+1.704/\rho$	$1.044/\rho$	$0.66+0.660/\rho$
		6	(ac,bd)	$0.66+1.968/\rho$	$1.211/\rho$	$0.66+0.757/\rho$
		7	(ad,bc)	$0.66+2.155/\rho$	$1.605/\rho$	$0.66+0.550/\rho$
3	6	1	(ab,c,d)	$0.66+1.381/\rho$*	$0.481/\rho$	$0.66+0.900/\rho$
		2	(ac,b,d)	$0.66+2.040/\rho$	$0.833/\rho$	$0.66+1.207/\rho$
		3	(ad,b,c)	$0.66+1.704/\rho$	$0.0297/\rho$*	$0.66+1.407/\rho$
		4	(a,bc,d)	$0.66+2.282/\rho$	$1.399/\rho$	$0.66+0.883/\rho$*
		5	(a,bd,c)	$0.66+1.668/\rho$	$0.378/\rho$	$0.66+1.290/\rho$
		6	(a,b,cd)	$0.66+2.063/\rho$	$0.563/\rho$	$0.66+1.500/\rho$
4	1	1	Current Valuation (a,b,c,d)	$0.66+1.740/\rho$*	0*	$0.66+1.740/\rho$*

* Denotes members of the efficient set and efficient frontier.
k = number of price indexes in the partition (valuation rule),
$S(n, k)$ = number of ways of partitioning a set of n distinct elements (four in our example) into k subsets (price indexes),
λ = partition identifier,
ρ = diversification parameter. See footnote 3,
$MSE(\tilde{R}_{k\lambda})$ = total mean squared error of valuation rule $\tilde{R}_{k\lambda}$,
$MV(\tilde{R}_{k\lambda})$ = movement error of valuation rule $\tilde{R}_{k\lambda}$, and
$MR(\tilde{R}_{k\lambda})$ = measurement error of valuation rule $\tilde{R}_{k\lambda}$.

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}. \tag{5}$$

These definitions of the family of valuation rules and a summary of their known econometric properties set the stage for an analysis of their properties in the presence of measurement errors.

II. Efficiency of Valuation Rules Under Price Movement and Measurement Errors

Two factors give rise to the error of valuation rules—price changes and errors in measurement of price changes. We shall label the errors arising from these factors *movement* (MV) and *measurement* (MR) errors, respectively. The magnitude of both types of valuation errors is influenced by the divergence between the relative weights of various assets in the portfolios of individual firms on one hand, and the effective relative weights assigned to their respective price changes by various valuation rules on the other. Greater divergence increases the movement error but decreases the measurement error. Under current valuation, each good is in a price index by itself, and the two sets of weights are identical. This identity of actual and effective weights reduces movement error to its minimum but takes the measurement error to its maximum. As fewer, more aggregated, price indexes are used in valuation, movement error increases; the measurement error, on the other hand, decreases as errors in measurement of individual price changes included within the same index tend to cancel each other. Econometrically efficient valuation rules are found by examining the effect of aggregation on the sum of these two types of errors.

The error of measurement in the price relatives of individual goods arises from several sources. Sampling error, substitution bias, and quality error cause a theoretical price relative to differ from its empirical measures.

Sampling is an important source of error in price indexes since indexes are estimated from samples, not from entire populations. The data for construction of price indexes are gathered almost entirely from a network of samples: samples of products, of localities, of reporters, and of points in time. Therefore, the value of a price index depends on the particular samples. For example, the Producer Price Index (PPI) is constructed by collecting approximately 10,000 quotations for 2,800 commodities in primary markets of the United States every month (Bureau of Labor Statistics 1976). Even though probability-proportional-to-size sampling technique cut the sampling error in the PPI as of August 1978, (Early 1978, 1979), error cannot be avoided entirely.

Since both the Consumer Price Index (CPI) and the PPI are based on a fixed-weight formula (Laspeyres Index) and thus substitution is not considered, there exists an inherent upward bias due to substitution: a price index may remain the same while real prices fall because of the arrival of cheaper substitutes of similar quality. Empirical work shows the substitution bias to be relatively small (Triplett 1975).

Another source of error in price indexes is the change in quality of goods. If quality improves over time, the index may have an upward bias. But Triplett (1975) showed that quality error is not necessarily upward sloping and the sign of quality error is not determined because price indexes are explicitly designed to attempt to adjust for quality changes. Since the optimum quality adjustment procedure is unknown, it is difficult to measure the quality error. Furthermore, many price quotations used to construct indexes are not subject to bargaining and thus are not always equal to actual transaction prices (Price Statistics Review Committee of the National Bureau of Economic Research 1961, 70).

We analyze the efficiency of valuation rules when errors caused by both price movement and price measurement are present. Let $\tilde{\underline{r}}$, the observed price relative vector, be the sum of unobserved true price relative $\underline{r}$ and measurement error $\underline{\epsilon}$: $\tilde{\underline{r}}=\underline{r}+\underline{\epsilon}$, where $E(\underline{r})=\underline{\mu}$, $\mathrm{Var}(\underline{r})=E(\underline{r}-\underline{\mu})(\underline{r}-\underline{\mu})'=\Sigma$, $E(\underline{\epsilon})=\underline{0}$, $\mathrm{Var}(\underline{\epsilon})=E(\underline{\epsilon}\underline{\epsilon}')=\Delta$, $\mathrm{Cov}(\underline{\epsilon},(\underline{r}-\underline{\mu}))=0$, and $\sim$ designates variables measured with error.[8]

[8] We assume throughout that measurement errors are unbiased; $E(\underline{\epsilon})=0$. None of the qualitative results in the article change when these errors have nonzero expectation of $\underline{\nu}$. The quantitative effect can be obtained by the following substitutions in the text: Δ by $(\Delta+\underline{\nu}\underline{\nu}')$, Δ_{uu} by $(\Delta_{uu}+\underline{\nu}_u\underline{\nu}_u')$, and $\underline{\delta}$ by $(\underline{\delta}+\dot{\underline{\nu}})$ where $\dot{\underline{\nu}}$ is the vector of squared elements of $\underline{\nu}$. Hall and Shriver (1990) document empirical evidence suggesting that prices in the PPI database may have an upward bias relative to a privately gathered database.

Following the notation used above, the mean squared difference between the estimator $\tilde{R}_{k\lambda}$ and the measurement-error-free current valuation free of measurement errors, $R_{n,1}$ is:[9]

$$MSE(\tilde{R}_{k\lambda}) = E_{\underline{w}} E_{\underline{r}} E_{\underline{\varepsilon}} (\tilde{R}_{k\lambda} - R_{n,1})^2 \tag{6}$$

$$= \left(1-\frac{1}{\rho}\right)\underline{\omega}'\Delta\underline{\omega} + \frac{1}{\rho}\sum_{u=1}^{k}\frac{\underline{\omega}_u'\Delta_{uu}\underline{\omega}_u}{\underline{\omega}'\underline{e}} + \frac{1}{\rho}\underline{\omega}'(\underline{\sigma}+\dot{\underline{\mu}}) - \frac{1}{\rho}\sum_{u=1}^{k}\frac{\underline{\omega}_u'(\Sigma_{uu}+\underline{\mu}_u\underline{\mu}_u')\underline{\omega}_u}{\underline{\omega}_u'\underline{e}},$$

where Δ_{uu} is a submatrix of Δ consisting of those rows and columns of Δ that correspond to elements included in the uth price index (see the appendix for derivation).

In the special case when the measurement error is zero (i.e., $\Delta = 0$), expression (6) is reduced to expression (2) because the first two terms of expression (6) drop out. Therefore, expression (2) can be seen as the movement error (MV) component of the error of the valuation rule, $\tilde{R}_{k\lambda}$. Similarly, when there is no price movement error (i.e., $\underline{\mu} = \underline{0}$ and $\Sigma = 0$), the last two terms of expression (6) drop out, leaving the measurement error (MR) component given by:

$$MR(\tilde{R}_{k\lambda}) = \left(1-\frac{1}{\rho}\right)\underline{\omega}'\Delta\underline{\omega} + \frac{1}{\rho}\sum_{u=1}^{k}\frac{\underline{\omega}_u'\Delta_{uu}\underline{\omega}_u}{\underline{\omega}_u'\underline{e}}. \tag{7}$$

Proposition 1 states the result that the total MSE of valuation rules can be decomposed into two additive components:

Proposition 1: The total mean squared error of a valuation rule is decomposable into the sum of its movement error (MV) and its measurement error (MR):

$$MSE(\tilde{R}_{k\lambda}) = MR(\tilde{R}_{k\lambda}) + MV(\tilde{R}_{k\lambda}). \tag{8}$$

An important property of measurement error is given by the following theorem.

Theorem 1: Measurement error (MR) of valuation rules increases monotonically with the fineness of the index system used in them.

Proof: See the appendix.

Since measurement error increases with the fineness of a valuation rule, it follows immediately that this error is highest for the n-index (i.e., current) valuation and lowest for the 1-index (i.e., GPL) valuation given by expressions (9) and (10), respectively:

$$MR(\tilde{R}_{n,1}) = \left(1-\frac{1}{\rho}\right)\underline{\omega}'\Delta\underline{\omega} + \frac{1}{\rho}\underline{\omega}'\underline{\delta} \tag{9}$$

and

$$MR(\tilde{R}_{1,1}) = \underline{\omega}'\Delta\underline{\omega}, \tag{10}$$

where $\underline{\delta}$ is the vector of diagonal elements of Δ. In addition, the historical cost valuation, being independent of $\underline{\tilde{r}}$, is entirely free of valuation errors caused by measurement errors in $\underline{\tilde{r}}$.

[9] Note that $R_{n,1}$ is the measurement-error-free current valuation or principal aggregate which is unobservable. In contrast $\tilde{R}_{n,1}$ is the current valuation estimate of $R_{n,1}$ based on observed price data.

Note that the effect of increasing the fineness of valuation rules on their measurement errors is opposite to its effect on their movement errors. Use of coarser price indexes reduces errors of measurement through diversification. However, coarse price indexes also make it more difficult to track the movement of prices. This is a key property of valuation rules and it plays a critical role in identifying the minimum *MSE* or efficient valuation rules.

Efficient Estimators

For any given value of k, there are a total of $S(n,k)$ distinct estimators or valuation rules. Let $\tilde{R}_{k^*}$ denote the efficient k-index estimator, such that it has the least mean squared error of all k-index valuation rules:

$$MSE(\tilde{R}_{k^*}) \leq MSE(\tilde{R}_{k\lambda}), \qquad \text{for all } \lambda = 1,2,\ldots,S(n,k). \tag{11}$$

Definition 1: The *Efficient set* of estimators consists of those estimators whose mean squared error is not greater than the mean squared error of any other estimator with the same number of price indexes. *Efficient frontier*, $H(k)$, $k=1,\ldots,n$, is the set of mean squared errors associated with the efficient set of estimators:

$$H(k) = MSE(\tilde{R}_{k^*}), \qquad k=1,\ldots,n. \tag{12}$$

For each k, the efficient set has one or more valuation rules because several k-index valuation rules may attain the minimum *MSE*. However, only one point corresponding to this *MSE* lies on the efficient frontier. In the illustrative numerical example, we use the following parameter values:

$$\underline{\mu}' = (0.1,\ 0.2,\ 0.6,\ 0.4),\ \Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \text{ and } \Delta = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Then the complete set of estimators for the four-good economy consists of 16 elements, as shown in table 1. From these numbers, it is easy to verify Proposition 1 (decomposability of total error into *MV* and *MR*), Theorem 1 (*MR*'s monotone increase in the fineness of the estimator), and that the GPL valuation has the lowest measurement error of all estimators other than the historical-cost valuation, while the current valuation has the highest measurement error. Partitions that constitute the efficient set and their corresponding mean squared errors on the efficient frontier are marked by asterisks in the fifth column of table 1.

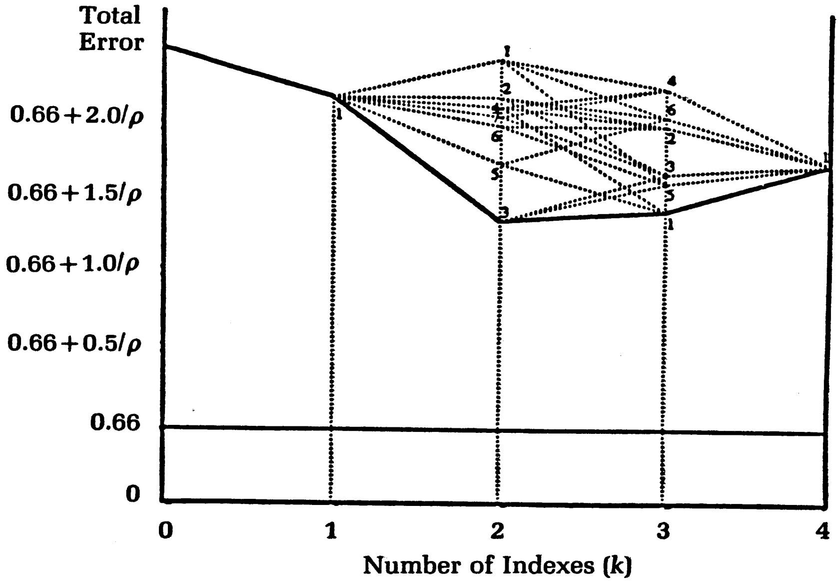
In the absence of measurement error ($\Delta=0$), the efficient frontier decreases monotonically in k (see Sunder and Waymire 1983). This monotonicity can be verified from column 6 of table 1 and the efficient frontier marked $\Delta=0$ in panel B of figure 1. This frontier in figure 1 is also convex in k ($1 \leq k \leq n$) although convexity has not yet been proved for the general case.

When movement error is zero ($\underline{\mu}=0$, $\Sigma=0$), it follows from Theorem 1 that the efficient frontier is monotonically increasing in k.[10] This monotonicity can also be con-

[10] Since $MR(\tilde{R}_{k\lambda})$ is monotonically increasing in fineness and since there always exists a $(k+1)$-partition that is strictly finer than any given k-partition, it follows that the *MSE* of estimators in the efficient set must be increasing in k when movement error is zero.

Figure 1
Mean Squared Error of Valuation Rules: Example 1*

Panel A. Total Error:



Panel B. Movement Error Only ($\Delta=0$):

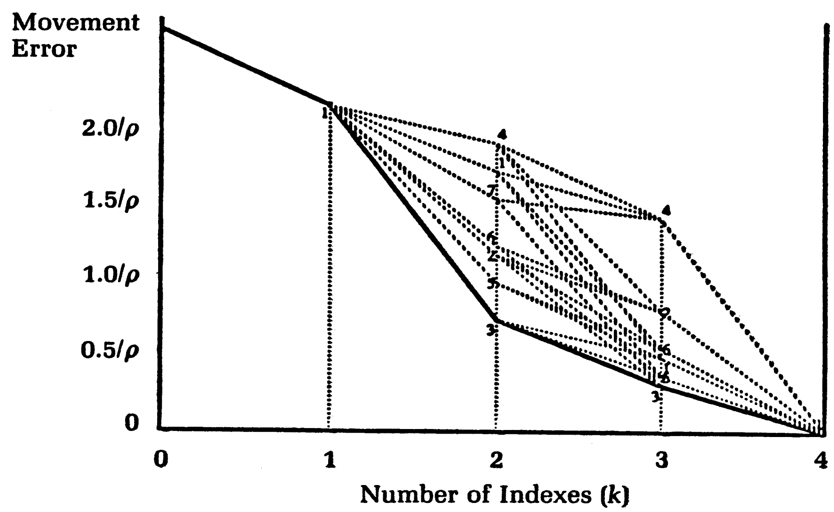


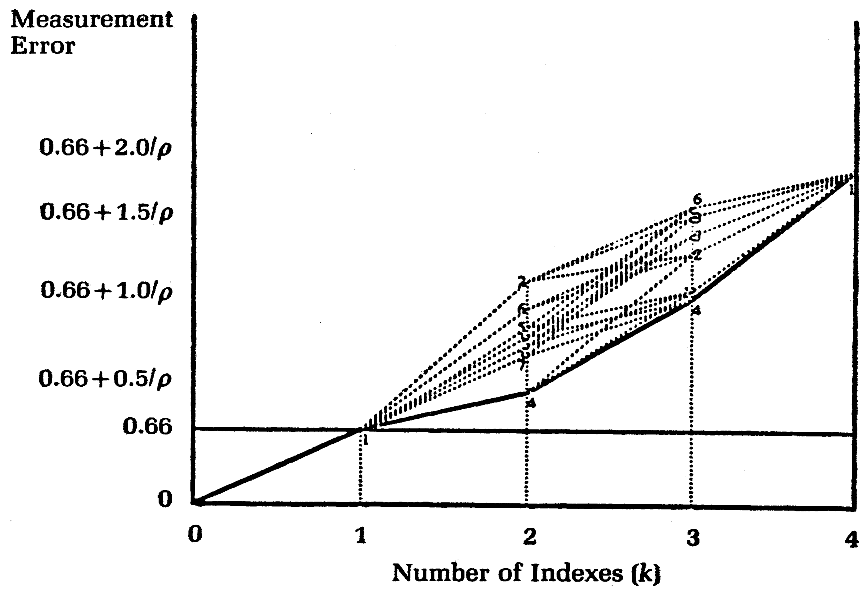
Figure 1 continues on next page.

firmed from column 7 of table 1 and the efficient frontier marked $\mu=0$, $\Sigma=0$ in panel C of figure 1. The efficient frontier in the figure is also convex in k $(1 \leq k \leq n)$ though convexity of the frontier in general remains to be proved.

In the general case when both movement and measurement errors are present, the efficient frontier is not necessarily the sum of the two efficient frontiers for each type of error. A valuation rule that is part of the efficient set for measurement errors is not necessarily in the efficient set for movement errors, and vice versa. A valuation rule can be in the efficient set for the total error even if it is not in the efficient set for either

Figure 1—*Continued*

Panel C. Measurement Error Only ($\mu\Sigma=0$):



* Each connecting line segment indicates fineness-coarseness relation between valuation rules.

type of error separately. For example, when $k=3$, minimum movement, measurement, and total errors are attained by three different valuation rules ((ad,b,c), (a,bc,d), and (ab,c,d), respectively) as can be seen in table 1 and figure 1. The efficient set and efficient frontier for total errors is also shown in table 1 and panel A of figure 1. This efficient frontier also is convex in k $(1\leq k<n)$ in the numerical example.

Convexity of the Efficient Frontier

If the efficient frontier for the total errors were proved to be convex in k, the task of finding the efficient frontier would be simplified considerably.[11] In all numerical examples we have been able to construct so far, the efficient frontier is found to be convex. Sunder and Waymire's (1983) search for an efficient frontier using the PPI data base yielded a highly convex estimate of the efficient frontier. Though the general proof of convexity has eluded us thus far, we have been able to prove the convexity of the efficient frontier in the special case when Σ and Δ are diagonal matrices and the economy-wide relative weights (ω_i) as well as the expected relative price changes (μ_i) across all goods in the economy are identical. Under these assumptions, we first find a method of constructing the efficient estimator of the current value for a given value of k; second, the efficient frontier corresponding to this set of estimators is proved to be convex; and third, the minimum of the efficient set (the global optimum estimator) is identified.

[11] Convexity of the efficient frontier for each type of error (movement and measurement) is insufficient to ensure the convexity of the efficient set for the total errors because the sum of two minimums is only a lower bound for the minimum of the sums.

Theorem 2: Let mean price relatives for all goods be equal ($\mu = m\underline{e}$, where m is a scalar and $\underline{e}$ is a vector of unit elements), relative weights of all goods in the economy be equal

$$(\underline{\omega} = \frac{1}{n}\underline{e}),$$

and Σ and Δ be diagonal matrices ($\sigma_{ij} = \delta_{ij} = 0$ for all $i \neq j$). Then the k-index efficient estimator of the current value is constructed by grouping those $(n-k+1)$ goods that have the $(n-k+1)$ smallest algebraic values of $(\sigma_{jj} - \delta_{jj})$ into a single price index and letting each of the other $(k-1)$ goods with the larger algebraic values of $(\sigma_{jj} - \delta_{jj})$ be alone each in a price index. The *MSE* of this valuation rule $\tilde{R}_{k^*}$ is given by:

$$H(k) = \frac{1}{n^2}\left(1 - \frac{1}{\rho}\right)\sum_{j=1}^{n}\delta_{jj} + \frac{1}{\rho n}\sum_{j=1}^{n}\sigma_{jj} - \frac{1}{\rho n}\left\{\sum_{j=1}^{k-1}\beta_j + \sum_{j=k}^{n}\frac{\beta_j}{n-k+1}\right\}, \tag{13}$$

where $\beta_j = (\sigma_{jj} - \delta_{jj})$.

Proof: See the appendix.

Theorem 2 suggests that goods with small variability in price movements (σ_{jj}) are candidates for being lumped with other such goods; goods with larger variability in price changes will contribute more to movement error unless they stand alone. In contrast, goods with larger measurement error (δ_{jj}) are the most attractive candidates for being grouped with others in an index in order to maximize the benefits of diversification through aggregation.

In our numerical example, the value of $\beta_j = (\sigma_{jj} - \delta_{jj})$ for the four goods is -2, -1, 4, and 0, respectively. According to Theorem 2, the most efficient two-index estimator can be constructed by including the three goods with the smallest algebraic values of β_j (a, b, and d) in one index and allowing the fourth good c to be in an index by itself. The most efficient three-index estimator can be constructed by including the two goods with the smallest algebraic values (a and b) in one index and allows goods c and d to be each in an index by themselves. Table 1 and figure 1 confirm that (c,abd) is the efficient two-index estimator and (ab,c,d) is the efficient three-index estimator with respect to total *MSE*.

Expression (13) in Theorem 2 specifies the efficient frontier in the presence of both movement and measurement errors. Note that the efficient frontier given in Shih and Sunder (1987) is a special case of expression (13) with $\Delta = 0$. In the cases covered by the theorem, the procedure for identifying estimators that are members of the efficient set is the same whether either type of error alone, or their sum, is being considered.

Theorem 3: The efficient frontier specified by expression (13) is convex.

Proof: See the appendix.

The following two conditions must be satisfied for the minimum of this convex function, $H(\bullet)$, to be attained at k^*:

Corollary 3.1: Under the assumptions of Theorem 3, the following two conditions are necessary and sufficient for the minimum of the efficient frontier to be attained at $k=k^*$:

$$\text{(a)}\quad \beta_{k^*}\leq\frac{-1}{(n-k^*)^2}\sum_{j=k^*+1}^{n}\beta_j \tag{14}$$

and

$$\text{(b)}\quad \beta_{k^*-1}\geq\frac{-1}{(n-k^*+1)^2}\sum_{j=k^*}^{n}\beta_j, \tag{15}$$

when goods have been arranged in the decreasing order of β_js.

Proof: See the appendix.

For the minimum of the efficient set to be attained at $k^*=1$, it is sufficient (but not necessary) that β_j be negative for all js. In other words, if the variance of measurement error exceeds the variance of movement error for all goods, the efficient frontier is monotonically increasing in k. A necessary and sufficient condition for attaining the minimum of the efficient frontier at $k^*=1$ (i.e., at GPL valuation) can be obtained by substitution in expression (14):

$$\beta_1\leq-\sum_{j=1}^{n}\frac{\beta_j}{(n-1)^2}, \tag{16}$$

where goods have been arranged such that $\beta_1\geq\beta_j$ for all $j>1$.

For the minimum to be attained at $k^*=n$, it is sufficient (but not necessary) that β_j be positive for all js. In other words, when the variance of price relatives exceeds the variance of measurement error for every good, the efficient frontier is monotonically decreasing in k. The necessary and sufficient condition for attaining the minimum of the efficient frontier at $k^*=n$ (i.e., at the current valuation) can be obtained by substitution in expression (15):

$$\beta_{n-1}\geq-\beta_n. \tag{17}$$

In summary, excluding historical-cost valuation, the GPL valuation provides the minimum *MSE* estimate of the unobserved current value of a firm's assets when condition (16) is met. This counterintuitive result is obtained when the variance of errors in measurement of prices are large compared with the variance of price relatives. Under these conditions, the advantages of aggregation from diversifying random errors of measurement dominate the disadvantages of using a coarser index set to estimate the current value.

Similarly, again excluding the historical-cost valuation, when condition (17) is met, the n-index system (with a price index for each good) is the minimum *MSE* estimator. This happens when the variance of measurement error is small relative to the variance of price relatives.

More generally, when conditions (14) and (15) of Corollary 3.1 are met, neither the GPL nor the current valuation is the minimum *MSE* estimator. Instead, the minimum *MSE* of all valuation rules is obtained by using price indexes at some intermediate level of aggregation, where the benefits and disadvantages of aggregation are balanced at the margin.

In the numerical example, conditions (14) and (15) are satisfied for an intermediate minimum but conditions (16) and (17) are not. To verify this, consider the four goods in decreasing order by the value of $\beta_j = (\sigma_{jj} - \delta_{jj})$, which is c(5−1=4), d(2−2=0), b(3−4=−1), and a(1−3=−2). According to condition (16), in order for the single-index estimator (GPL) valuation) to be the minimum *MSE* estimator, the value of $(\sigma_{jj} - \delta_{jj})$ for good c, 4, would have to be less than or equal to

$$-\frac{0-1-2}{3^2},$$

which is not true. In order for the global minimum *MSE* estimator to be attained at $k=4$ (current valuation), it is sufficient that β_j be positive for every good, and this condition is not satisfied for goods a and b. The necessary and sufficient condition (17) requires that the value of $(\sigma_{jj} - \delta_{jj})$ for good b, (−1), be greater than or equal to the negative value for good a, −(−2). This condition is not satisfied either. Conditions of (14) and (15) for the minimum of the efficient frontier are satisfied at $k^*=2$ in the numerical example. This can also be confirmed from table 1 and figure 1.

III. Global Minimum Mean Squared Error Estimator

In the absence of errors of measurement, the *MSE* of the historical cost estimator is necessarily greater than the error of all other estimators. The top panel of figure 2 shows the relationship of the error associated with historical-cost valuation to the error of the GPL valuation ($k=1$), the current valuation ($k=n$), and other members of the efficient set of estimators that use more than one but less than n price indexes. The error of the historical-cost estimator exceeds the GPL error by $\underline{\omega}'(\Sigma + \underline{\mu}\underline{\mu}')\underline{\omega}$, and the GPL error exceeds the current valuation error by:

$$\frac{1}{\rho}\{\underline{\omega}'(\underline{\sigma}+\dot{\underline{\mu}}) - \underline{\omega}'(\Sigma + \underline{\mu}\underline{\mu}')\underline{\omega}\}.$$

When price measurement errors are present but price movement errors are absent, the relationship of the historical cost valuation (*HC*), the GPL valuation, the current valuation (*CV*), and other estimators are shown in the second panel of figure 2. The GPL error exceeds the *HC* error (which is zero) by $\underline{\omega}'\Delta\underline{\omega}$. The *CV* error, in turn, exceeds the GPL error by:

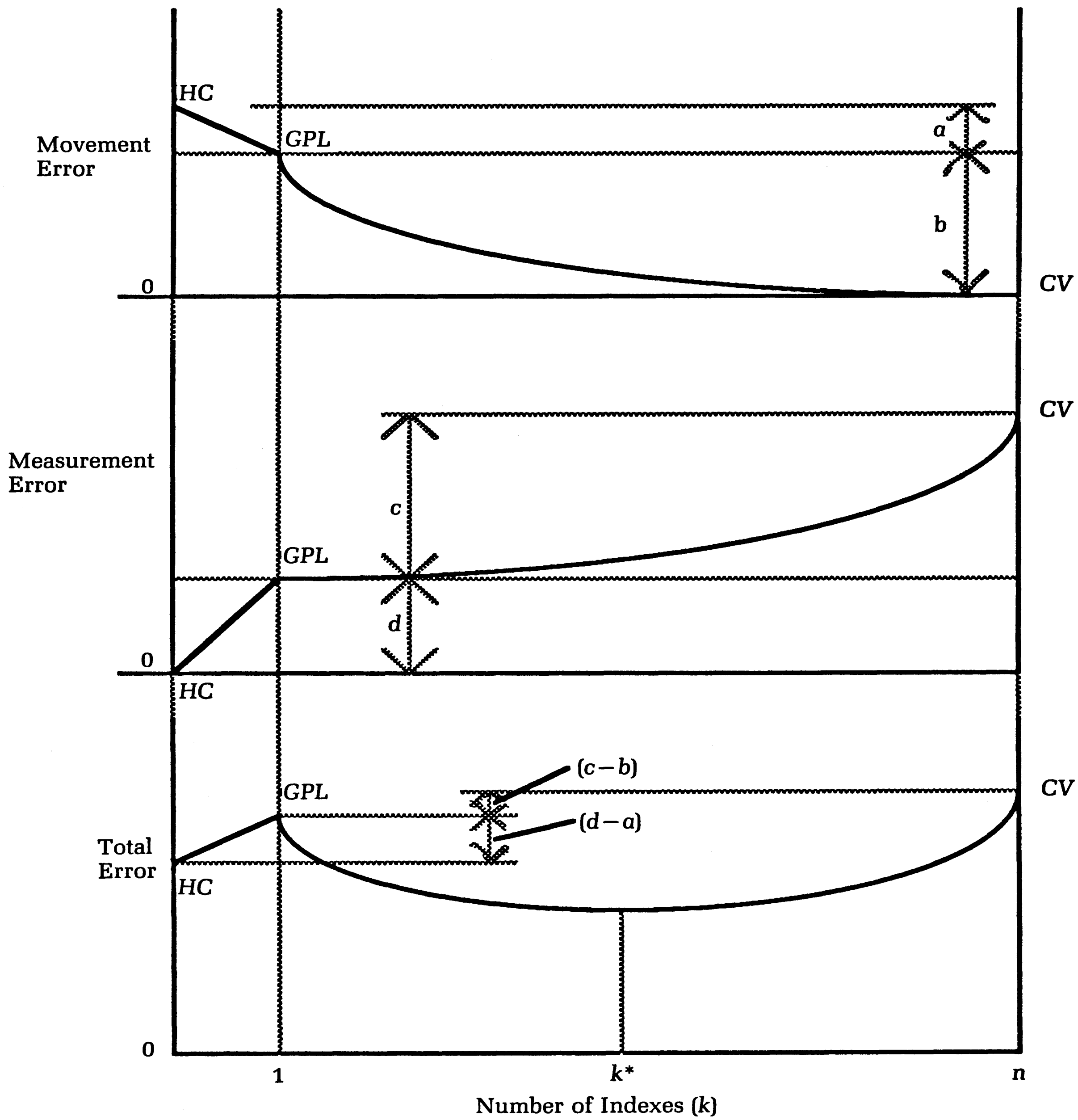
$$\frac{1}{\rho}(\underline{\omega}'\underline{\delta} - \underline{\omega}'\Delta\underline{\omega}).$$

In Proposition 1, we have already shown that the price measurement and movement errors of estimators are additive. Accordingly, Theorem 4 states the conditions for the historical-cost valuation to be a more accurate estimate of the current economic value of the firm's assets than the GPL or the *CV* estimators, respectively.

Theorem 4:

(i) The historical cost valuation dominates the general price level valuation in accuracy if and only if:

Figure 2
Mean Squared Error of the Historical Cost Valuation Versus the Other Valuation Rules*



* HC = Historical Cost Valuation,
GPL = General Price Level Valuation,
CV = Current Valuation,
a = Movement Error (HC) − Movement Error (GPL) $= \underline{\omega}'(\Sigma + \underline{\mu}\underline{\mu}')\underline{\omega}$,

b = Movement Error (GPL) − Movement Error (CV) $= \frac{1}{\rho}\{\underline{\omega}'(\underline{\sigma} + \underline{\mu}) - (\Sigma + \underline{\mu}\underline{\mu}')\underline{\omega}\}$,

c = Measurement Error (CV) − Measurement Error (GPL) $= \frac{1}{\rho}(\underline{\omega}'\underline{\delta} - \underline{\omega}'\Delta\underline{\omega})$, and

d = Measurement Error (GPL) − Measurement Error (HC) $= \underline{\omega}'\Delta\underline{\omega}$.

$$\underline{\omega}'(\Delta-\Sigma+\underline{\mu}\underline{\mu}')\underline{\omega}>0.$$

(ii) The historical cost valuation dominates the current valuation in accuracy if and only if:

$$\left(1-\frac{1}{\rho}\right)\underline{\omega}'(\Delta-\Sigma-\underline{\mu}\underline{\mu}')\underline{\omega}+\frac{1}{\rho}\underline{\omega}'(\underline{\delta}-\underline{\sigma}-\underline{\dot{\mu}})>0.$$

Proof: See the appendix.

The historical-cost valuation can dominate not only the GPL valuation and the current valuation in accuracy, it can also dominate the minimum *MSE* valuation rule identified in Corollary 3.1. The numerical example given above is modified to illustrate this dominance (example 2). Let $\underline{\omega}$, $\underline{\mu}$, and Σ remain unchanged and allow the variance-covariance matrix of measurement errors, Δ, to be 2.6 times as large as previously assumed:

$$\Delta=\begin{bmatrix} 7.8 & 0 & 0 & 0 \\ 0 & 10.4 & 0 & 0 \\ 0 & 0 & 2.6 & 0 \\ 0 & 0 & 0 & 5.2 \end{bmatrix}.$$

Then the *MSE* for Example 2 are as follows:

Historical-cost valuation: $1.260+\dfrac{2.214}{\rho}$,

General price level valuation: $1.716+\dfrac{2.214}{\rho}$,

Current value valuation: $1.716+\dfrac{4.524}{\rho}$, and

Minimum *MSE* estimator (*HC* valuation): $1.260+\dfrac{2.214}{\rho}$.

It is easily seen that the historical-cost estimator provides, in this plausible example, the closest approximation of the current economic value of firms as shown in table 2 and figure 3. Also note that this dominance is valid for all values of the diversification parameter ρ. As the mean and variance of price changes, $\underline{\mu}$ and Σ, increase, it becomes less likely that the historical cost valuation will dominate other estimators.

IV. Concluding Remarks

Extant theories of valuation, largely deterministic in nature, can be integrated into a single framework to facilitate direct comparisons of their statistical properties in specified economic environments. In this study, we have shown that when prices change and price data are subject to errors of measurement, neither the current-valuation rule nor the general-price-level valuation rule is necessarily the minimum *MSE* estimator of the unobserved economic value of baskets of assets. Instead, the most accurate valuation is likely to be attained by using specific price indexes at an appropri-

Table 2
Properties of Estimators of Current Value in a Four-Good Economy: Example 2

1	2	3	4	5	6	7
				Economy-Wide Average Errors		
k	$S(n,k)$	Serial Number	Partition	$MSE(\tilde{R}_{k\lambda})$ =	$MV(\tilde{R}_{k\lambda})$ +	$MR(\tilde{R}_{k\lambda})$
0	—	—	Historical	$1.260+2.214/\rho$*	$1.260+2.214/\rho$*	0*
1	1	1	GPL(abcd)	$1.716+2.214/\rho$*	$2.214/\rho$*	1.716*
2	7	1	(a,bcd)	$1.716+3.364/\rho$	$1.765/\rho$	$1.716+1.599/\rho$
		2	(b,acd)	$1.716+3.724/\rho$	$1.205/\rho$	$1.716+2.519/\rho$
		3	(c,abd)	$1.716+2.246/\rho$*	$0.756/\rho$*	$1.716+1.490/\rho$
		4	(d,abc)	$1.716+2.564/\rho$	$1.911/\rho$	$1.716+0.653/\rho$*
		5	(ab,cd)	$1.716+2.760/\rho$	$1.044/\rho$	$1.716+1.716/\rho$
		6	(ac,bd)	$1.716+3.179/\rho$	$1.211/\rho$	$1.716+1.968/\rho$
		7	(ad,bc)	$1.716+3.035/\rho$	$1.605/\rho$	$1.716+1.430/\rho$
3	6	1	(ab,c,d)	$1.716+2.821/\rho$*	$0.481/\rho$	$1.716+2.340/\rho$
		2	(ac,b,d)	$1.716+3.971/\rho$	$0.833/\rho$	$1.716+3.138/\rho$
		3	(ad,b,c)	$1.716+3.955/\rho$	$0.297/\rho$*	$1.716+3.658/\rho$
		4	(a,bc,d)	$1.716+3.695/\rho$	$1.399/\rho$	$1.716+2.296/\rho$*
		5	(a,bd,c)	$1.716+3.732/\rho$	$0.378/\rho$	$1.716+3.354/\rho$
		6	(a,b,cd)	$1.716+4.463/\rho$	$0.563/\rho$	$1.716+3.900/\rho$
4	1	1	Current Valuation (a,b,c,d)	$1.716+4.524/\rho$*	0*	$1.716+4.524/\rho$*

* Denotes members of the efficient set and efficient frontier.
k = number of price indexes in the partition (valuation rule),
$S(n,k)$ = number of ways of partitioning a set of n distinct elements (four in our example) into k subsets (price indexes),
λ = partition identifier,
ρ = diversification parameter. See footnote 3,
$MSE(\tilde{R}_{k\lambda})$ = total mean squared error of valuation rule $\tilde{R}_{k\lambda}$,
$MV(\tilde{R}_{k\lambda})$ = movement error of valuation rule $\tilde{R}_{k\lambda}$, and
$MR(\tilde{R}_{k\lambda})$ = measurement error of valuation rule $\tilde{R}_{k\lambda}$.

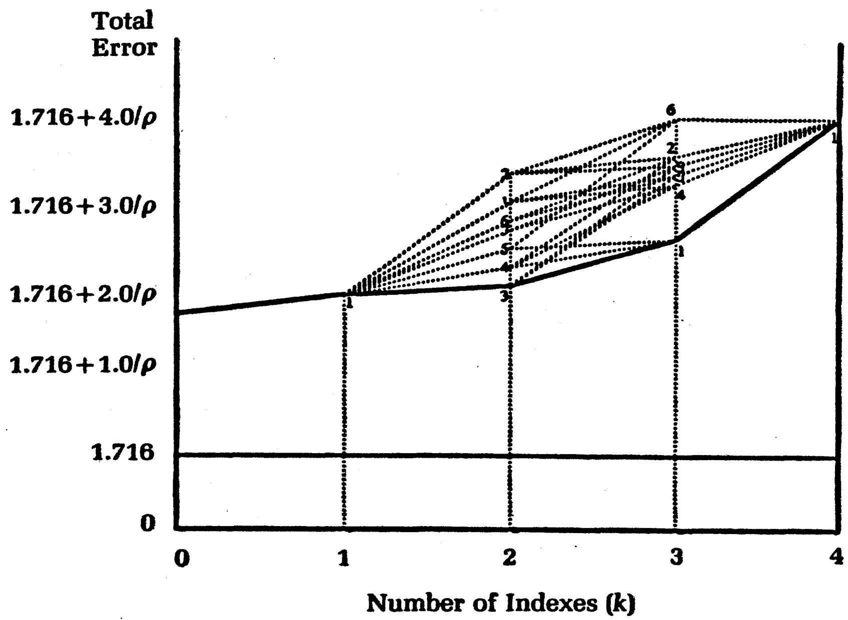
ate level of aggregation. Many firms implemented *SFAS 33* using specific price indexes; it would be interesting to find how the levels of aggregation chosen by individual firms relate to the theoretically efficient levels.

We have shown that if the error of measurement in current prices is sufficiently large, the historical-cost valuation may provide a statistically more accurate approximation of the unobserved economic value of assets than is provided by the current valuation. We have also derived conditions under which the historical-cost valuation is a more accurate estimate of the unobserved current economic value of assets than the most accurate of all linear valuation rules that use price indexes. Apparently, which valuation rule provides the best estimate of the current economic value is not a matter of theory or principle, but simply a matter of the relative magnitudes of parameters that characterize the economy: relative weights of various goods in the economy ($\underline{\omega}$), expected percentage price change for individual goods ($\underline{\mu}$), and the covariance matrices of these price changes (Σ) and of measurement errors in price changes (Δ).

These results have several testable implications. Other things being equal, valuation

Figure 3
Mean Squared Error of Valuation Rules: Example 2*

Panel A. Total Error:



Panel B. Movement Error Only (Δ=0):

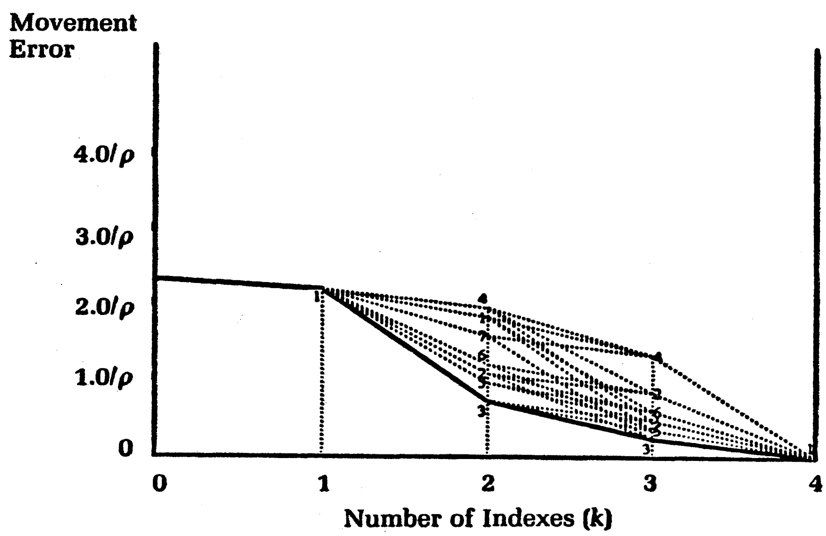
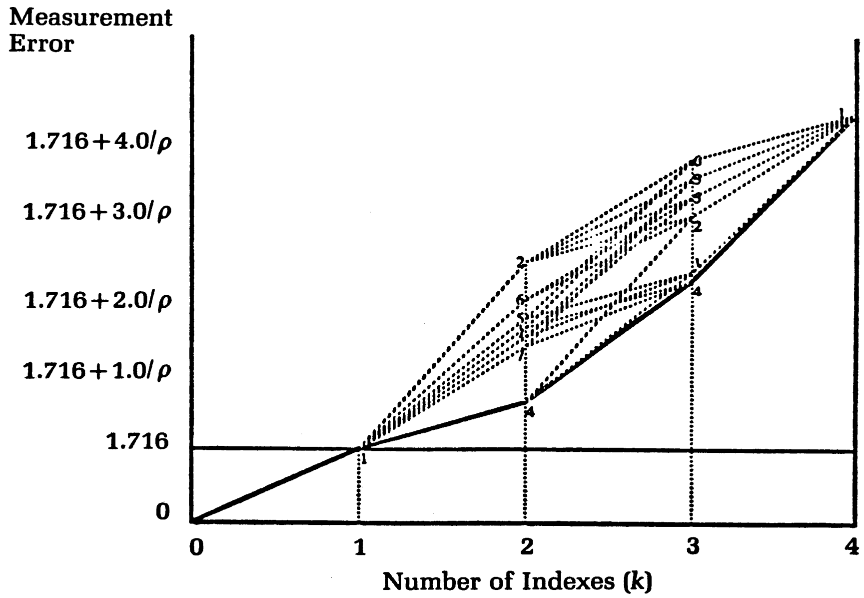


Figure 3 continues on next page.

rules based on a finer set of price indexes will be more informative for those firms and industries: (1) whose assets have a larger mean rate of price change; (2) whose assets have a more variable rate of price change; and (3) whose assets are traded in relatively perfect and complete markets, permitting more accurate measurement of change in their price. Real estate, oil and gas deposits, films, videos, software, and patents are

Figure 3—*Continued*

Panel C. Measurement Error Only ($\mu=0$, $\Sigma=0$):



* Each connecting line segment indicates fineness-coarseness relation between valuation rules.

examples of assets whose prices may have large measurement errors. Current valuation of firms with large holdings of such assets may not be more accurate than their historical valuation. Most of the empirical work has focussed on cross-sectional analysis so far (see Beaver et al. 1980, 1982; Gheyara and Boatsman 1980; Ro 1980). Our model suggests that closer attention to the economic environments of specific firms and industries will increase the power of tests to detect the information value of current value data reported by firms.

Second, the model suggests that efficient valuation rules may be quite different for different firms and industries. During the same period of time, historical cost may be most informative for some industries, while GPL, a 10- or 100- index valuation rule may be the most informative for others.

Third, the level of aggregation at which asset values are adjusted to their current estimates has a major impact. Since both *ASR 190* and *SFAS 33* granted wide latitude to individual firms in choosing the level of aggregation, different firms may have reported at quite different levels of aggregation, making it difficult to draw conclusions from cross-sectional studies that ignore this heterogeneity.

Fourth, the degree to which extant accounting valuation practices approximate the efficient rules can be determined empirically by (1) estimating the numerical values of parameters of the model, (2) identifying efficient valuation rules, and (3) comparing such rules with the extant practice. Theorem 2 provides a guide to testing whether the Bureau of Labor Statistics' scheme of clustering prices of individual goods into price indexes is an efficient one. Broadly speaking, the theoretical framework for valuation rules will make it possible to map the characteristics of valuation rules more precisely, and to conduct empirical tests of propositions about these characteristics.

Fifth, during periods of inflation, increasing the revaluation interval is likely to increase the magnitude of $\Sigma + \underline{\mu}\underline{\mu}'$ (expectation of squared price relatives) faster than the magnitude of Δ (expectation of squared measurement errors in price relatives) if measurement errors are more likely to be diversified away across time periods. Under these conditions, current valuation is more likely to dominate over historical and GPL over longer intervals of persistent inflation (or deflation).

We have used a number of assumptions about parameters and the model to derive these results. Some of these assumptions (e.g., those used in Theorem 2 to prove convexity of the efficient frontier) are mere stopgap measures until somebody is able to prove (or disprove) the convexity of this function under more general conditions. Other assumptions are based on our belief that they allow us to obtain important qualitative results from a simple model without undue violence to the environment we seek to understand. Others may wish to modify such assumptions to capture the implications of various refinements in the basic model.

For example, we asssume that the asset portfolio of all individual firms has identical distribution. Lim and Sunder (1990) drop this assumption and model an industry-segmented economy to examine the properties of valuation rules that utilize industry specific versus economywide price indexes. We also assume that the measurement errors in price relatives are unbiased. There is some empirical evidence (see Swanson and Shriver 1987; Hall and Shriver 1990; Swanson 1990) that certain price databases may have an upward bias. The model can be modified to explore the impact of this and other such refinements (see fn. 8). We have assumed that the price data for all assets of all firms in the economy are included in constructing the price indexes used for valuation. Additional valuation errors could be generated when this assumption is relaxed in various ways. For example, it is rarely possible to sample price changes for *all* assets in an economy; even the Bureau of Labor Statistics resorts to sampling for only about 3,000 different goods and services. Beaver et al. (1982, fn. 2) point out that changes in value of unrecorded assets goes unreported, and, as a practical matter, even some of the recorded assets may be excluded from revaluation. Further work would be needed to assess the magnitude of effect such deviations may have on predictions of the simple model presented here.

There may be situations for which the expected error and expected squared error loss functions used in this study may be considered inadequate. Computer simulations could be used to discover which of our results would generalize to other loss functions.

Finally, the analysis here is limited to linear valuation rules. Lim (1990) derives properties of nonlinear (e.g., lower-of-cost-or-market) valuation rules using a similar framework.

Appendix

I. Derivation of the Mean Squared Error Expression (6).

$$MSE(\tilde{R}_{k\lambda})=E_{\underline{w}}E_{\underline{r}}E_{\underline{\epsilon}}(\tilde{R}_{k\lambda}-R_{n,1})^2=E_{\underline{w}}E_{\underline{r}}E_{\underline{\epsilon}}\{(\underline{\omega}^{k\lambda})'\underline{\tilde{r}}-\underline{w}'\underline{r})^2$$

where:

$$\omega_j^{k\lambda}=\frac{w_j^*}{\omega_j^*}\omega_j,$$

w_j^* = total weight in $\underline{w}$ of goods included in the same index as good j,
ω_j^* = total weight in $\underline{\omega}$ of goods included in the same index as good j,
$\underline{\omega}_u$ = vector of ω_js included in the uth index, and
Σ_{uu} and Δ_{uu} are submatrices of Σ and Δ, respectively, consisting of rows and columns corresponding to the goods included in the uth index,

$$=E_{\underline{w}}E_{\underline{r}}E_{\underline{\epsilon}}\{(\underline{\omega}^{k\lambda})'\underline{\tilde{r}}\underline{\tilde{r}}'\underline{\omega}^{k\lambda}+\underline{w}'\underline{r}\underline{r}'\underline{w}-2(\underline{\omega}^{k\lambda})'\underline{\tilde{r}}\underline{r}'\underline{w}\}. \tag{A.1}$$

The right hand side of expression (A.1) is the expectation of the sum of the terms of an nxn symmetric matrix. The jth diagonal element of this matrix is:

$$(r_j+\epsilon_j)^2\left(\frac{w_j^*\omega_j}{\omega_j^*}\right)^2+w_j^2r_j^2-2(r_j+\epsilon_j)r_j\frac{w_j^*\omega_j}{\omega_j^*}w_j, \tag{A.2}$$

where $\epsilon_j=\tilde{r}_j-r_j$.

Using the mutually independent distributions of $\underline{w}$, $\underline{r}$, and $\underline{\epsilon}$ specified earlier, the expectation of the jth diagonal term is given by:[12]

$$\sigma_{jj}\left(\frac{\omega_j}{\rho}-\frac{\omega_j^2}{\rho\omega_j^*}\right)+\mu_j^2\left(\frac{\omega_j}{\rho}-\frac{\omega_j^2}{\rho\omega_j^*}\right)+\delta_{jj}\left\{\frac{\omega_j^2}{\rho\omega_j^*}+\left(1-\frac{1}{\rho}\right)\omega_j^2\right\}, \tag{A.3}$$

where ρ is the number of multinomial draws used to construct $\underline{w}$.

The i, jth off-diagonal element of the matrix, if both goods i and j are included in the same index, is given by:

$$(r_i+\epsilon_i)(r_j+\epsilon_j)w_i^*w_j^*\frac{\omega_i\omega_j}{\omega_i^*\omega_j^*}+w_iw_jr_ir_j-2\frac{w_i^*}{\omega_i^*}\omega_iw_j(r_i+\epsilon_i)r_j \tag{A.4}$$

and its expectation with respect to $\underline{w}$, $\underline{r}$, and $\underline{\epsilon}$ is given by:

$$-\sigma_{ij}\frac{\omega_i\omega_j}{\rho\omega_i^*}-\mu_i\mu_j\frac{\omega_i\omega_j}{\rho\omega_i^*}+\delta_{ij}\left\{\frac{\omega_i\omega_j}{\rho\omega_i^*}+\left(1-\frac{1}{\rho}\right)\omega_i\omega_j\right\}. \tag{A.5}$$

Finally, the i, jth off-diagonal element of the matrix if goods i and j are not included in the same index under valuation rule $\tilde{R}_{k\lambda}$ is also given by expression (A.4) and its expectation is:

$$\delta_{ij}\left(1-\frac{1}{\rho}\right)\omega_i\omega_j. \tag{A.6}$$

Adding up all terms of this matrix with some rearrangement yields the mean squared error (MSE) of a valuation rule $\tilde{R}_{k\lambda}$ in the presence of measurement error:

$$MSE(\tilde{R}_{k\lambda})=\left(1-\frac{1}{\rho}\right)\underline{\omega}'\Delta\underline{\omega}+\frac{1}{\rho}\sum_{u=1}^{k}\frac{\underline{\omega}_u'\Delta_{uu}\underline{\omega}_u}{\underline{\omega}'\underline{e}}+\frac{1}{\rho}\underline{\omega}'(\underline{\sigma}+\underline{\hat{\mu}})-\frac{1}{\rho}\sum_{u=1}^{k}\frac{\underline{\omega}_u'(\Sigma_{uu}+\underline{\mu}_u\underline{\mu}_u')\underline{\omega}_u}{\underline{\omega}_u'\underline{e}}, \tag{A.7}$$

II. Proof of Theorems

Proof of Theorem 1.

The first term in equation (7) is a constant with respect to various index configurations. To show that the second term is monotonically increasing in fineness, the proof by Sunder (1978, 364–5) for movement error is

[12] See footnote 3 for distribution of $\underline{w}$.

directly applicable because Δ is a positive semidefinite symmetric matrix. Proof of Theorem 1 follows immediately.

Q.E.D.

Proof of Theorem 2.

From equation (6), mean squared error of valuation rule $\tilde{R}_{k\lambda}$ is:

$$MSE(\tilde{R}_{k\lambda})=\frac{1}{n^2}\left(1-\frac{1}{\rho}\right)\sum_{j=1}^{n}\delta_{jj}+\frac{1}{\rho n}\left\{\sum_{j=1}^{n}(\sigma_{jj}+m^2)-\sum_{u=1}^{k}\left(\frac{1}{n_u}\sum_{j=1}^{n_u}\beta_j+n_u m^2\right)\right\} \tag{A.8}$$

$$=\frac{1}{n^2}\left(1-\frac{1}{\rho}\right)\sum_{j=1}^{n}\delta_{jj}+\frac{1}{\rho n}\sum_{j=1}^{n}\sigma_{jj}-\frac{1}{\rho n}\sum_{j=1}^{n}\frac{\beta_j}{n_j}$$

where n_u is the number of goods in the uth index and n_j is the number of goods in the index in which good j is included. In equation (A.8), all terms except:

$$\sum_{j=1}^{n}\frac{\beta_j}{n_j}$$

are constant with respect to index configurations; therefore minimizing *MSE* is equivalent to maximizing:

$$\sum_{j=1}^{n}\frac{\beta_j}{n_j}.$$

Let:

$$\sum_{j=1}^{n}\frac{\beta_j}{n_j}=A_{k\lambda}$$

for the valuation rule $\tilde{R}_{k\lambda}$ and the best $A_{k\lambda}$ given k be denoted by A_k^*. Without loss of generality, let the n goods be ordered such that $\beta_j \geq \beta_{j+1}$, $j=1,2,\ldots,(n-1)$. Since the n-index system is unique:

$$A_n^*=A_{n,1}=\sum_{j=1}^{n}\beta_j. \tag{A.9}$$

For $k=n-1$, some two goods, indexed p and q must be combined into a single index while all other $(n-2)$ goods form single-good indexes.

$$A_{n-1,1}=\sum_{\substack{j=1\\ j\neq p,q}}^{n}\beta_j+\frac{\beta_p+\beta_q}{2}=\sum_{j=1}^{n}\beta_j-\frac{\beta_p+\beta_q}{2}. \tag{A.10}$$

Since the first term on the right hand side is a constant, to maximize $A_{n-1,1}$, the sum, $\beta_p+\beta_q$, should be minimized; that is, two goods with the smallest algebraic values of β_j should be clubbed together into a single index. Since β_n and β_{n-1} are the two smallest values by the above ordering,

$$A_{n-1}^*=\sum_{j=1}^{n}\beta_j-\frac{\beta_n+\beta_{n-1}}{2}. \tag{A.11}$$

When $k=n-2$, there are two possible ways of partitioning the above $(n-1)$ indexes into $(n-2)$ indexes:

(1) Combine a third good, indexed r, into the above two-good index which contains goods n and $(n-1)$. Let its accuracy measure be $A_{n-2,1}^1$.
(2) Combine other two single-good indexes for goods indexed s and t, into one. Let its accuracy measure be $A_{n-2,1}^2$.

Then we can show: $A_{n-2}^{1*}>A_{n-2}^{2*}$.

$$A_{n-1,1}^1=\sum_{\substack{j=1\\ j\neq n,n-1,r}}^{n}\beta_j-\frac{\beta_n+\beta_{n-1}+\beta_r}{3}=\sum_{j=1}^{n}\beta_j-\frac{2(\beta_n+\beta_{n-1}+\beta_r)}{3}. \tag{A.12}$$

Since β_{n-2} is the third smallest, $A^{1}_{n-2,1}$ is maximized by setting $r=n-2$:

$$A^{1*}_{n-2}=\sum_{j=1}^{n}\beta_j-\frac{2(\beta_n+\beta_{n-1}+\beta_{n-2})}{3}. \tag{A.13}$$

For the second method of constructing $(n-2)$ indexes from n goods,

$$A^{2}_{n-2,1}=\sum_{\substack{j=1\\ j\neq n,n-1,s,t}}^{n}\beta_j+\frac{\beta_n+\beta_{n-1}+\beta_s+\beta_t}{2}. \tag{A.14}$$

$A^{2}_{n-2,1}$ is maximized by setting $s=n-2$ and $t=n-3$. Therefore,

$$A^{2*}_{n-2}=\sum_{\substack{j=1\\ j\neq n,n-1,n-2,n-3}}^{n}\beta_j+\frac{\beta_n+\beta_{n-1}+\beta_{n-2}+\beta_{n-3}}{2}. \tag{A.15}$$

In order to determine which of these two methods yields a better $(n-2)$-index system, we calculate their difference:

$$\begin{aligned}A^{1*}_{n-2}-A^{2*}_{n-2}&=\frac{1}{2}(\beta_n+\beta_{n-1}+\beta_{n-2}+\beta_{n-3})-\frac{2}{3}(\beta_n+\beta_{n-1}+\beta_{n-2})\\ &=\frac{1}{6}\{2\beta_{n-3}-(\beta_n+\beta_{n-1})+(\beta_{n-3}-\beta_{n-2})\}\\ &\geq 0,\end{aligned}$$

since $\beta_j\geq\beta_{j+1}$ for all js. Therefore, the good with the third smallest β_j should be combined with the two-good index in A^{*}_{n-1} in order to construct the most efficient $(n-2)$-index valuation rule.

If the best $(n-2)$-index valuation is obtained by combining the three goods with the three smallest algebraic values of β_j into a single index, it can similarly be shown that the best $(n-3)$-index valuation requires combining the four goods with the four smallest values of β_js. Thus a mathematical induction proves Theorem 3. Equation (13) in the theorem follows from (A.8) and (A.13).

Q.E.D.

Proof of Theorem 3.

From Theorem 2, the optimum index configurations for $k=m$, $m-1$, and $m+1$ are as follows:

$$A^{*}_{m}=\sum_{j=1}^{m-1}\beta_j+\sum_{j=m}^{n}\frac{\beta_j}{n-m+1}, \tag{A.16}$$

$$A^{*}_{m-1}=\sum_{j=1}^{m-2}\beta_j+\sum_{j=m-1}^{n}\frac{\beta_j}{n-m+2},$$

and

$$A^{*}_{m+1}=\sum_{j=1}^{m}\beta_j+\sum_{j=m+1}^{n}\frac{\beta_j}{n-m}.$$

Now for the convexity, we will show:

$$H(m)-H(m-1)\leq H(m+1)-H(m)$$

or

$$H(m)-\frac{1}{2}\{H(m-1)+H(m+1)\}\leq 0. \tag{A.17}$$

From expressions (A.8) and (A.16), left hand side of expression (A.17):

$$\leq\frac{1}{\rho n}\left\{-\left(\sum_{j=1}^{m-1}\beta_j+\sum_{j=m}^{n}\frac{\beta_j}{n-m+1}\right)+\frac{1}{2}\left(\sum_{j=1}^{m-2}\beta_j+\sum_{j=1}^{m}\beta_j+\sum_{j=m-1}^{n}\frac{\beta_j}{n-m+2}+\sum_{j=m+1}^{n}\frac{\beta_j}{n-m}\right)\right\}$$

$$=\frac{1}{2\rho n}[(\beta_m-\beta_{m-1})+\frac{1}{(n-m)(n-m+1)(n-m+2)}\{2\sum_{j=m+1}^{n}\beta_j-2(n-m)(n-m+2)\beta_m$$
$$+(n-m)(n-m+1)(\beta_{m-1}+\beta_m)\}].$$

Since $\sum_{j=m+1}^{n}\beta_j\leq(n-m)\beta_j$,

left hand side

$$\leq\frac{1}{2\rho n}[(\beta_m-\beta_{m-1})+\frac{1}{(n-m)(n-m+1)(n-m+2)}\{2(n-m)\beta_j+(n-m)(-n+m-3)\beta_m$$
$$+(n-m)(n-m+1)\beta_{m-1}\}]$$
$$=\frac{1}{2\rho n}\left\{(\beta_m-\beta_{m-1})-\frac{\beta_m-\beta_{m-1}}{n-m+2}\right\}$$
$$\leq 0,$$

which is true because $\beta_m\leq\beta_{m-1}$ and $(n-m+2)\geq 1$ for all ms. This proves the convexity of the efficient frontier.

Q.E.D.

Proof of Corollary 3.1.

Since $H(k)$ is convex in k, for its minimum to be attained at $k=k^*$, it is necessary and sufficient that:

$$H(k^*-1)\geq H(k^*) \text{ and } H(k^*)\leq H(k^*+1).$$

Conditions of Corollary 4.1 are derived directly by substitution from expression (13) in the above inequalities.

Q.E.D.

Proof of Theorem 4.

(i) From expression (A.7), *MSE* of the general price level valuation is:

$$MSE(\tilde{R}_{1,t})=\underset{\sim}{\omega}'\Delta\underset{\sim}{\omega}+\frac{1}{\rho}\{\underset{\sim}{\omega}'(\underset{\sim}{\sigma}+\dot{\underset{\sim}{\mu}})-\underset{\sim}{\omega}'(\Sigma+\underset{\sim}{\mu}\underset{\sim}{\mu}')\underset{\sim}{\omega}\}. \qquad (A.18)$$

From equations (4) and (A.18),

$$MSE(\tilde{R}_{1,t})-MSE(R_{0,t})=\underset{\sim}{\omega}'(\Delta-\Sigma-\underset{\sim}{\mu}\underset{\sim}{\mu}')\underset{\sim}{\omega}.$$

For the historical cost valuation to dominate the general price level valuation, this difference must be positive.

(ii) From equations (4) and (6), the difference between *MSE* of the historical-cost valuation and that of the current valuation is given as in Theorem 4.

Q.E.D.

References

Apostol, T. 1967. *Calculus*. Vol. II. Wâltham, MA: Blaisdell.

Beaver, W. H., A. Christie, and Paul A. Griffin. 1980. The information content of SEC Accounting Series Release No. 190. *Journal of Accounting & Economics* 2 (August): 127–57.

———, and J. Demski. 1979. The nature of income measurement. *The Accounting Review* 54 (January): 38–46.

———, P. A. Griffin, and W. R. Landsman. 1982. The incremental information content of replacement cost earnings. *Journal of Accounting and Economics* 4 (January): 15–39.

Bureau of Labor Statistics. 1976. *Handbook of Methods,* Bulletin 1910. Washington, D.C., Government Printing Office.

Chambers, R. J. 1966. *Accounting, Evaluation and Economic Behavior.* Englewood Cliffs, NJ: Prentice-Hall.

Early, J. 1978. Improving the measurement of producer price changes. *Monthly Labor Review* 101 (April): 7–15.

———. 1979. The producer price index revision: Overview and pilot survey results. *Monthly Labor Review* 102 (December): 11–19.

Eckern, S., and R. Wilson. 1974. On the theory of the firm in an economy with incomplete markets. *Bell Journal of Economics and Management Science* (Spring): 171–80.

Edwards, E. O., and P. W. Bell. 1961. *The Theory and Measurement of Business Income.* Berkeley: University of California Press.

Feller, William. 1968. *An Introduction to Probability Theory and Its Applications.* Vol. I, 3d ed. New York: Wiley.

Financial Accounting Standards Board. 1979. *Statement of Financial Accounting Standards No. 33: Financial Reporting and Changing Prices.* Stamford, CT: FASB.

Gheyara, K., and J. Boatsman. 1980. Market reaction to the 1976 replacement cost disclosures. *Journal of Accounting and Economics* 2 (August): 107–25.

Hall, T. W. 1982. An empirical test of the effect of asset aggregation on valuation accuracy. *Journal of Accounting Research* 20 (Spring): 139–51.

———, and Keith A. Shriver. 1990. Econometric properties of asset valuation rules under price movement and measurement errors: An empirical test. *The Accounting Review* 65 (July): 537–56.

Ijiri, Y. 1967. *The Foundations of Accounting Measurement: A Mathematical, Economic and Behavioral Inquiry.* Englewood Cliffs, NJ: Prentice-Hall.

———. 1968. The linear aggregation coefficient as the dual of the linear correlation coefficient. *Econometrica* 36 (April): 252–59.

———, and J. Noel. 1984. A reliability comparison of the measurement of wealth, income, and force. *The Accounting Review* 59 (January): 52–63.

Lev, Baruch. 1969. *Accounting and Information Theory. Studies in Accounting Research #2.* Sarasota, FL: American Accounting Association.

Lim, S. S. 1990. *Accuracy of partial valuation rules and optimal use of price indexes.* Ph. D. dissertation. University of Minnesota, Minneapolis.

———, and S. Sunder. 1990. Accuracy of linear valuation rules in industry segmented environments: Industry vs. economy weighted indexes. *Journal of Accounting and Economics* 13 (July): 167–88.

The Price Statistics Review Committee of National Bureau of Economic Research. 1961. *The Price Statistics of the Federal Government,* General Series No. 73.

Radner, R. 1974. A note on unanimity of stockholders' preferences among alternative production plans: A reformulation of the Eckern-Wilson model. *Bell Journal of Economics and Management Science* 5 (Spring): 181–84.

Ro, B. 1980. The adjustment of security prices to the disclosure of replacement cost accounting information. *Journal of Accounting and Economics* 2 (August): 159–89.

Shih, S., and S. Sunder. 1987. Design and tests of an efficient search algorithm for accurate linear valuation system. *Contemporary Accounting Research* 4 (Fall): 16–31.

Shriver, K. 1986. Further evidence on the marginal gains in accuracy of alternative levels of specificity of the producer price indexes. *Journal of Accounting Research* 24 (Spring): 151–65.

———. 1987. An empirical examination of the potential measurement error in current cost data. *The Accounting Review* 62 (January): 79–96.

Sterling, R. 1970. *Theory of the Measurement of Enterprise Income.* Lawrence: University Press of Kansas.

Sunder, S. 1976. Properties of accounting numbers under full costing and successful-efforts costing in the petroleum industry. *The Accounting Review* 51 (January): 1–18.

———. 1978. Accuracy of exchange valuation rules. *Journal of Accounting Research* 16 (Autumn): 341–67.

———, and G. Waymire. 1983. Marginal gains in accuracy of valuation from increasingly specific price indexes: Empirical evidence for the U.S. Economy. *Journal of Accounting Research* 21 (Autumn): 565–80.

———, and ———. 1984. Accuracy of exchange valuation rules: Additivity and unbiased estimation. *Journal of Accounting Research* 22 (Spring): 396–405.
Swanson, E. 1990. Relative measurement errors in valuing plant and equipment under current cost and replacement cost. *The Accounting Review* 65 (October): 911–24.
———, and K. A. Shriver. 1987. The accounting for changing prices experiment: A valid test of usefulness? *Accounting Horizons* 1 (September): 69–78.
Tippett, M. 1987. Exchange valuation rules: Optimal use of specific price indexes in asset valuation. *Accounting and Business Research* 17 (Spring): 141–54.
Triplett, J. 1975. The measurement of inflation: A survey of research on the accuracy of price indexes. P. Earl (ed.). *Analysis of Inflation.* Lexington, MA: Lexington Books. 19–82.
Tritschler, C. 1969. Statistical criteria for asset valuation by specific price index. *The Accounting Review* 44 (January): 99–123.