Design and tests of an efficient search algorithm for accurate linear valuation systems*

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Abstract. A valuation system partitions the set of goods to be valued into multiple disjoint subsets and the current value of the goods is estimated via price indexes covering these subsets. Efficient valuation systems yield a relatively small economy-wide average of mean squared errors with respect to the true total current cost of the goods. Several algorithms have been designed to search for efficient valuation systems. These algorithms, however, do not take advantage of the information contained in the characteristic parameters of the goods to be valued. We present the design and test of a search algorithm that is substantially more efficient than those in the literature. The relative efficiency of the algorithm is gained through the use of information contained in the weights, the expected values, and the variance-covariance structure of the price changes of the goods.

Résumé. Un système d'évaluation subdivise l'ensemble des biens à être évalués en plusieurs sous-ensembles disjoints et la valeur actuelle des biens est estimée grâce à des indices de prix couvrant ces sous-ensembles. Des systèmes d'évaluation efficaces produisent, pour l'ensemble de l'économie, une moyenne relativement faible des erreurs moyennes au carré, par rapport au coût actuel réel total des biens. Plusieurs algorithmes ont été conçus pour découvrir des systèmes d'évaluation efficaces. Toutefois, ces algorithmes n'intègrent pas l'information que renferment les paramètres caractérisant les biens à évaluer. Nous présentons la conception et le test d'un algorithme de recherche qui s'avère considérablement plus efficace que ceux mentionnés dans les recherches antérieures. L'efficacité relative de cet algorithme est obtenue grâce à l'intégration de l'information contenue dans les pondérations, les valeurs espérées et la structure de variance-covariance des fluctuations de prix des biens.

Introduction

Sunder and Waymire (1983) designed an algorithm to identify relatively accurate valuation systems. Their algorithm exploited the relationship between the fineness of a valuation system and its accuracy. This paper presents the design and tests of an algorithm which is substantially more efficient than the Sunder and Waymire (1983) algorithm in searching for accurate valuation systems. The gain in efficiency and speed is accomplished by utilizing the information contained in the

* We have benefited from comments received at the University of Minnesota Accounting workshop and from an anonymous referee. Financial support for the research was provided by the Accounting Research Center and the Honeywell and the McKnight Foundations.

Contemporary Accounting Research Vol. 4 No. 1 pp 16–31

variance-covariance matrix and the mean vector of relative price changes of individual assets.

The design of an asset valuation system that uses a given number of price indexes but sacrifices as little of the accuracy of valuation as possible is the subject of this paper. There are many ways of partitioning a set of, say, n assets into k subsets ($k \leq n$) to form k price indexes. Unless the change in the price of each asset is separately measured (that is, a separate price index is constructed exclusively for each asset) some accuracy is inevitably lost in the aggregation process. Sunder (1978) showed that, using the economy-wide average of the mean squared error of valuation (AMSE) for firms (as an inverse measure of accuracy), some aggregation systems yield more accurate valuations than others; i.e. they sacrifice less accuracy in the process of aggregation.

The total number of ways in which a set of n goods can be partitioned into k nonempty, nonoverlapping subsets to construct k price indexes is given by Stirling Numbers of the Second Kind. These numbers are extremely large even for moderate values of n and k.[1] An exhaustive search of all possible partitions to discover the most efficient partition is too expensive even for modern computers and generous research budgets. Standard techniques of optimization using calculus are not helpful in searching for efficient valuation rules over discrete partitions of sets. Hence, the need arises for a satisfactory search algorithm.

Identification of the efficient set is important for both theoretical and practical reasons. The practical significance of discovering efficient valuation rules is obvious: closer approximation of economic value in the sense of reduced error variance yields better economic decisions.[2] A valuation rule is an experiment in the sense of Blackwell (1953) when it gives information about the current value w′r. Other things being equal, the smaller the conditional variance of a valuation rule, the more informative or sufficient in the sense of Blackwell is the experiment. The new algorithm we propose in this paper is able to find valuation rules whose mean squared error is about 30 percent less than that of the rules found by the Sunder and Waymire (1983) algorithm.

From a theoretical standpoint we need to investigate the properties, especially the convexity, of the efficient set. Given the difficulties of analytical approach, empirical identification of this set provides an attractive alternative route. Design of fast search algorithms is essential for this purpose.

1 The number of distinct ways of partitioning a set of n distinct elements into k ($k \leq n$) nonempty subsets is given by Stirling Numbers of the Second Kind:

$$S(n, k) = \sum_{j=0}^{k-1} (-1)^j (k - j)^n / \{j!(k - j)!\}.$$

For example, there are 140 ways of forming five indexes from seven goods, 1050 ways from eight goods, 6951 ways from nine goods and 42,525 ways from ten goods.

2 One may be tempted to conclude from recent empirical studies that current valuation does not provide economic gains to shareholders. Lim and Sunder (1985) show that in most economic environments, current valuation of assets is not the closest approximation of economic value. When errors of measurement are present, even historical valuation may provide a closer approximation of economic value than current valuation does.

The next section of the paper defines the notation and the environment, followed by a section in which an investigation of how the economy-wide average of the mean squared error of a valuation rule depends on various parameters. The insights gained through this analysis are given in the form of four theorems which are discussed in the third section. These insights are used to design the search algorithm which is the topic of the third section. The testing procedures outlined and the results of comparing the speed and effectiveness of the new algorithm with that of the Sunder-Waymire algorithm are presented in the fourth section. Concluding remarks are offered at the end of the paper.

Accuracy of linear valuation systems

Consider an economy with n distinct goods or assets in quantities given by vector q and prices given by vectors p^0 and p^1 at times 0 and 1 respectively. The relative abundance of these goods, measured by their dollar value at time 0, in the economy is given by vector ω such that $\omega_i = q_i p_i^0 / qp^0$. The vector of relative price changes for n goods is r where $r_i = (p_i^1 - p_i^0)/p_i^0$. *Ex ante*, r is a random vector with expectation μ and variance-covariance matrix Σ. Let Γ be the expectation of rr′; i.e. $\Gamma = E(rr') = \Sigma + \mu\mu'$.

Let an individual firm in the economy be constructed by making an arbitrary number, say ρ, multinomial independent random draws with replacement from the economy's basket of goods. The relative abundance of goods in this randomly drawn basket, w, has a multinomial distribution with mean ω. Each firm in the economy is characterized by its vector of relative proportions, w.

The relative change in the value of firm w from time 0 to 1 is w′r. We denote this change by R_{cv}. This "true" change in value can be approximated by other valuation rules that use one or more price indexes.

Other valuation systems can be designed by constructing one or more price indexes for various (k) subsets of n assets and applying these indexes to the valuation of corresponding assets in each firm's basket of goods, w. For example, a two-index valuation system is created by partitioning the n assets into two nonempty subsets and calculating the average relative price change for assets in each subset by using the economy-wide relative weights, ω. These two average relative price changes, or price indexes, when applied to the goods in the firm's basket yield an estimate of the relative price change in the value of the whole basket.

There are many possible ways of forming two nonempty subsets of $n(n > 2)$ assets.[3] If there are a total of $S(n, 2)$ ways of doing so, we can create a valuation system corresponding to each one of these $S(n, 2)$ partitions. Let these valuation systems or partitions be indexed by i, $i = 1, 2, \ldots S(n, 2)$. We shall use symbol R_{2i} to denote the two-index valuation system which uses the i^{th} of these $S(n, 2)$ partitions. More generally, R_{ki} denotes the i^{th} of the $S(n, k)$ possible k-index valuation systems. There is only one way each of creating one index and n indexes respectively from n assets: $S(n, 1) = S(n, n) = 1$. For each intermediate value of k,

3 See footnote 1 above.

there are many different valuation systems because $S(n, k) > 1$. The purpose of this paper is to devise a way of identifying which of these $S(n, k)$ valuations for any given number k yields a relatively more accurate approximation of current valuation R_{cv}.

How well the valuation system R_{ki} approximates R_{cv} can be measured by the squared difference between these two numbers, $(R_{ki} - R_{cv})^2$. Since this measure depends on random vectors w and r, a more tractable question is the size of the squared distance on average. Taking expectations of the squared difference with respect to the probability distributions of r and w, Sunder (1978) showed that the average of mean squared error (AMSE) of R_{ki} is given by:

$$\text{AMSE}(R_{ki}) = (1/\rho)\left(\omega'\gamma - \sum_{u=1}^{k} ((\omega_u'\Gamma_{uu}\omega_u)/\omega_u' e)\right) \tag{1}$$

where:

e $=$ vector of 1's of an appropriate size;

ω $=$ n-vector of relative weights of n assets in the economy;

ω_u $=$ n_u-vector of relative weights of the assets in index u;

μ $=$ n-vector of expected relative price changes for n assets;

μ_u $=$ n_u-vector of expected relative price changes for the assets in index u;

Γ_{uu} $=$ $n_u \times n_u$ submatrix of Γ corresponding to n_u assets included in the u^{th} index;

γ $=$ diagonal vector of matrix Γ, of length n;

k $=$ the number of price indexes used in the valuation rule;

ρ $=$ the number of multinomial trials by which the bundle of assets for individual firms is randomly drawn with replacement from the economy-wide bundle.

Since AMSE is an inverse measure of accuracy of a k-index valuation system, the task is to find that k-index valuation system which has the minimum AMSE. Since the set of k-index valuation systems is closed and bounded, such a minimum exists. Let R_k^* denote the k-index valuation system that minimizes the AMSE. Then AMSE (R_k^*) is the lower bound of AMSE (R_k). There is one such lower bound for each value of k, $1 \leq k \leq n$. The n lower bounds constitute the efficient frontier.

Search for the efficient frontier

The number of ways in which n goods can be partitioned into k subsets $(1 < k < n)$ is relatively large even for a moderate value of n. Identifying the efficient frontier requires that for each value of k $(1 < k < n)$, $S(n, k)$ different valuation rules be evaluated and compared to discover the one which has the minimum AMSE. The design of a search algorithm is made more difficult by the fact that this search must be conducted over discrete partitions instead of a continuous domain and the usual optimization methods of calculus are not applicable. For any realistic value of n, an exhaustive search of all possible valuation systems is not practical even with fast computers and generous research budgets. The extant search algorithms leave much to be desired in efficiency and effectiveness. These algorithms follow either random or fineness search strategies. A random search algorithm uses a random number generator to partition the n goods into k subsets randomly. Each partition is examined and the one that generates the smallest AMSE is recorded. This is a relatively inefficient brute force strategy because it does not exploit any information about the properties of the valuation systems.

AMSE of valuation systems necessarily decreases with fineness[4] of the valuation system (see Sunder (1978) for proof). Sunder and Waymire (1983) utilized this property to devise an algorithm, which they called the systematic search algorithm. The algorithm starts out by randomly searching to identify a tentative frontier and then uses the fineness property to successively improve upon the tentative frontier. At any stage of execution, the algorithm concentrates its search for more efficient valuation rules among those partitions which are strictly finer or strictly coarser than the neighboring partitions on the currently identified tentative efficient frontier. For example, if at some point in execution of the algorithm, $\Pi_{k-1,i}$ and $\Pi_{k+1,j}$ are the best $(k - 1)$ and $(k + 1)$ partitions respectively found (i.e., are members of the tentative efficient frontier at that stage), search for an improved k-partition is concentrated among partitions which are strictly finer than $\Pi_{k-1,i}$ and those that are strictly coarser than $\Pi_{k+1,j}$. Sunder and Waymire (1983, 1984) demonstrated the relative efficiency of this search procedure over the random search strategy and reported their estimates of the efficient frontier based on the Producer Price Index database for 199 commodities. However, even the fineness search leaves too much to chance and does not exploit the information contained in parameters ω, Σ and μ. We shall call our strategy the "parametric search" algorithm because it takes advantage of the information contained in the parameters.

Parametric search

In order to exploit the properties of parameters ω (the vector of relative weights), μ (the vector of expected relative price changes), and Σ (the covariance matrix of relative price changes) to find the efficient frontier, their effect on the AMSE must be examined. The results of our investigation of these effects are summarized informally in the following paragraphs. Formal statements of the four theorems and their proofs are given in the Appendix. These theorems form the basis of our parametric search algorithm that exploits some of the information contained in the parameters.

Theorem 1 (Using diagonal elements of Σ): Goods with more volatile price changes should be separated from others and from one another in order to form price indexes for more accurate valuation systems.

4 Partition Π_1 is finer than partition Π_2 if and only if each subset in Π_1 is included in some subset in Π_2. For example, partition {(a), (b), (cd)} is finer than {(ab), (cd)} but is not comparable in fineness with {(ac), (bd)}.

Theorem 2 (Using μ): Goods with close expected relative price changes should be bundled together into a single index; goods with very different expected relative price changes should not be placed together in the same index in order to form price indexes for more accurate valuation systems.

Theorem 3 (Using ω): Goods with small weights should be bundled together into a single index and goods with large weights should be placed in an index each in order to form price indexes for more accurate valuation rules.

Theorem 4 (Using off-diagonal elements of Σ): Transfer of good j from an index containing set u of n_u goods to an index containing set v of n_v goods is more likely to improve the efficiency of valuation if:

i the average covariance of good j with goods in subset v is high;
ii the average covariance of good j with goods in subset u is low;
iii $n_v + 1 > n_u$ and the variance σ_{jj} is high;
iv $n_v + 1 < n_u$ and the variance σ_{jj} is low.

Theorems 1, 2, and 3 suggest that in designing a k-index valuation system, other things being equal, those goods that have relatively large variances, relatively large or small expected price changes, or relatively large weights, should have their exclusive one-good indexes if the covariances are all zero. The intuition behind these theorems is that those goods that are relatively abundant or whose price changes are "volatile" or "out of line" with the rest should not be bundled with other goods because such bundling will have a large adverse effect on the overall accuracy of the valuation system. Bundling will not cause too much distortion if, other things being equal, the goods that are bundled have relatively stable price changes, have mean price changes close to one another or carry relatively low weights. When price movements of goods are correlated, Theorem 4 suggests that subsets of goods that have high variances or covariances should be bundled together. This also makes intuitive sense because under these conditions the prices of these goods are likely to move in the same direction and to the same extent. Such bundling will not distort the valuation of any of the goods too much even though each of these goods has a relatively large variance. Bundling one of these high-variance goods with other low-variance goods, on the other hand, reduces the accuracy of the valuation system.

The new algorithm

In the following description of the algorithm, indexes i, j and k are used as follows: $i = 1, 2, \ldots n$ for the i^{th} good in the economy; $j = 1, 2, \ldots k$ for the j^{th} subset of n goods out of k subsets; $k = 1, 2, \ldots n$ the number of nonempty subsets into which n goods are partitioned.

Step 1 Start with $k = 1$. (There is only one way of partitioning n goods into 1 subset.)

Step 2 Take the first subset in the current partition, $j = 1$. Let this subset be denoted by J and its cardinality by n_J.

Step 3 Calculate the average value of the diagonal elements of Γ that correspond to the current subset.

$$\alpha = \left(\sum_{i \in J} \gamma_{ii} \right) \Big/ n_J.$$

Step 4 Identify a subset J' of J such that for each element of J', the diagonal element of Γ exceeds $f \cdot \alpha$, where f is an arbitrarily chosen constant greater than 1.

Step 5 Take the first element of J', indexed j' and identify all goods i, $i \in J$, $i \neq j'$, such that $\gamma_{j,i} \geq f \cdot \alpha$. Let the set of these goods be $J''(j')$ and let $\omega(j')$ be the total weight of elements of this set in ω.

Step 6 Repeat step 5 for each element of J'. Identify good j', and set $J''(j')$ for which $\omega(j')$ is the largest.

Step 7 Split subset J into two parts: $J''(j')$ and $J - J''(j')$. Use index $(k + 1)$ for the first subset, and retain index j for the remainder.

Step 8 Using Theorem 4, examine each element of subset j to check if transferring this element to subset $(k + 1)$ will result in a gain in efficiency. Transfer those elements which result in a gain in efficiency. The resulting partition which contains $(k + 1)$ subsets is the current partition.

Step 9 Calculate the AMSE of the $(k + 1)$ index valuation rule corresponding to the current partition. If it dominates the most accurate $(k + 1)$ index valuation rule identified to this point in the algorithm, replace the latter by the former in the memory.

Step 10 Repeat steps 2 through 9 for each $j = 1, 2, \ldots k$ to search for superior $(k + 1)$ index valuation rules. Let Π^*_{k+1} be the partition which has the smallest AMSE.

Step 11 Repeat steps 1 to 10 using $k = 2, 3, \ldots n - 1$. In step 2, start out by using the Π^*_{k+1} identified at the end of step 10 in the previous cycle.

Testing procedures

We analyzed the Producer Price Indexes (PPI) at the two-digit level and the three-digit level. Only those indexes for which all data from 1967 to 1979 are available are included in the analysis. All 15 indexes at the 2-digit level and 78 out of 85 indexes at the 3-digit level satisfied this requirement.

The 13-year time series (13 yearly observations) for each price index was transformed into relative change or return form yielding 12 observations. The sample estimates of the mean and the covariance matrix were calculated from these 12 observations, first for the 15 indexes at the two-digit level and then for the 78 indexes at the three-digit level. These sample estimates of μ and Σ were treated as parameter values in testing the search algorithm and the sampling errors were ignored. The relative weights, ω, at the two-digit level were those given by the Bureau of Labor Statistics of the U.S. Department of Labor (BLS, 1978). The relative weights at the three-digit level are provided by the BLS for 85 indexes. Because we used only 78 of these 85 indexes, these weights were rescaled to add up to unity for the 78 indexes.

Both the Sunder-Waymire (S-W) algorithm and the parametric search (PS) algorithm were coded in Turbo Pascal (Version 3.0) to run on an IBM personal computer and in PASCAL to run on a Cyber 70 main frame computer.[5] Tests of the two algorithms at the two-digit level were conducted using a personal computer; tests at the three-digit level needed the help of a main frame.

We used two statistics to compare the efficiency of the two algorithms. The first was for the amount of time needed by each algorithm to find equally efficient sets of valuation rules. The second was for the average ratio of the AMSEs of the most efficient valuation rules discovered by each algorithm when they were allowed to run on a computer for equal amounts of time.

Results

Two-digit level

The parametric search algorithm took 40 seconds on the IBM personal computer. The AMSE of the most accurate valuation rule for each value of k, ($k = 2, 3, \ldots 14$) is plotted as the lower of the two curves in Figure 1. The upper curve plots the AMSEs of the most accurate valuation rules found by the S-W algorithm in a 40-second run.[6] The parametric search algorithm clearly outperforms the S-W algorithm. The average of the ratio of ordinates of the parametric search and S-W accuracy functions is 0.65. In other words, given equal time, the parametric search procedure was able to find valuation rules whose AMSEs were, on average, only 65 percent of the AMSEs of the most accurate valuation rules found by the S-W algorithm.

Table 1 provides data on how well the S-W algorithm performed when allowed to run longer than 40 seconds. Its efficiency slightly exceeded the efficiency of the parametric search (101 percent) only when it was allowed to run for 400 seconds, or ten times the time used in parametric search.

Three-digit level

Figure 2 compares the AMSEs of the most accurate valuation rules found by each of the two algorithms for $k = 2, 3, \ldots 77$ in 26 second runs on Cyber 70. When the S-W algorithm was allowed to run for 258 seconds, the valuation systems it found were still less accurate than what the parametric search found in 26 seconds. The mean of the ratio of the ordinates is 0.85.

5 Both of these codes are available from the authors upon request.

6 Since the S-W algorithm depends partly on random search, we made four 40-second runs of this algorithm. The accuracy function given is the average of the four runs.

Figure 1 Efficiency of search algorithms PPI two-digit level

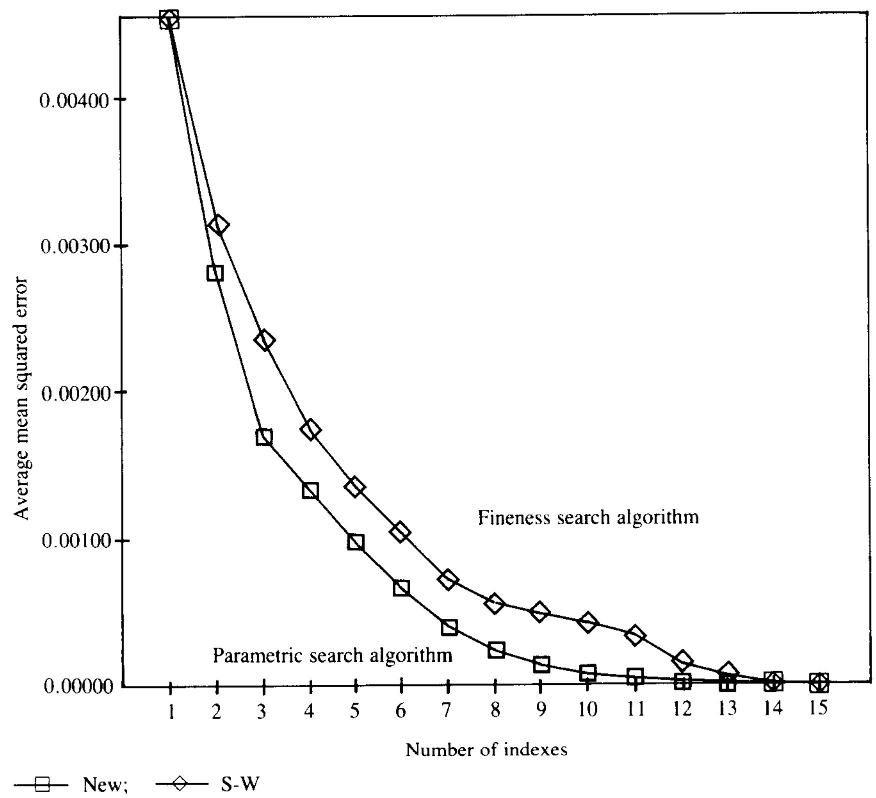


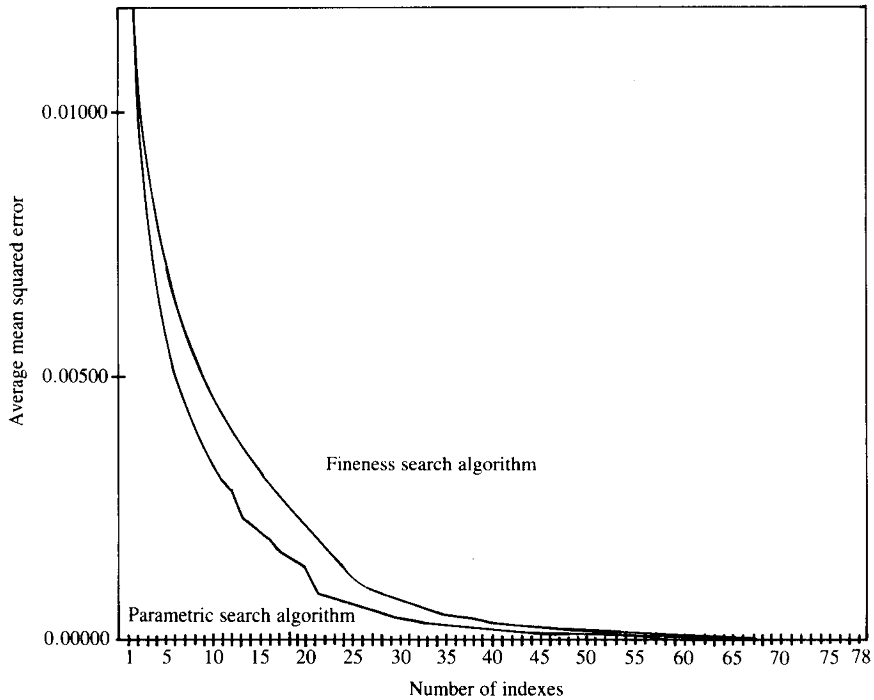
TABLE 1
Efficiency of the S-W algorithm relative to parametric search

Time allowed for S-W algorithm in seconds	40	90	180	270	400
Average of the ratio of ordinates of parametric search to S-W algorithm accuracy functions	0.65	0.85	0.92	0.96	1.01

Concluding remarks

Utilization of parametric information about μ, Σ and ω allows us to construct price index sets which are, relative to the output of other search techniques, more efficient in approximating the current values of firms. Parametric search dominates fineness search, which, in turn, dominates random search. We see our effort as only one step toward developing the techniques of devising more efficient

Figure 2 Efficiency of search algorithms PPI three-digit level



valuation rules. It seems possible to find ways to utilize the parametric information more efficiently than we have done.

After relatively efficient sets of price indexes are constructed, these sets will naturally be compared to those produced by the Bureau of Labor Statistics. These indexes, available off-the-shelf, are perhaps the most widely used in the economy. Are these indexes constructed efficiently for the purpose of valuation? Is it possible to improve upon the way the commodities are partitioned to produce the BLS index sets? We suspect the answer is yes but must withhold judgment until a careful examination is carried out.

Appendix: Proofs for theorems

The average mean square error for valuation rule R_{ki} is

$$\text{AMSE}\,(R_{ki}) = (1/\rho)\left\{\omega'\gamma - \sum_{u=1}^{k} (\omega_u'/\omega_u' e)(\Sigma_{uu} + \mu_u\mu_u')\omega_u\right\}. \tag{A.1}$$

Since ρ and $\omega'\gamma$ are both unchanged by index configurations, these terms can be ignored in making comparisons of accuracy of valuation systems based on the same parameters ω, μ, and Σ. The second term of (A.1) is enough to make all comparisons. Let

$$A(R_{ki}) = \sum_{u=1}^{k} \frac{\omega_u'(\Sigma_{uu} + \mu_u\mu_u')\omega_u}{\omega_u'e} . \tag{A.2}$$

AMSE can be minimized by maximizing $A(R_{ki})$.

Theorem 1:
Let Σ be a diagonal matrix (i.e. price changes of all goods are uncorrelated with one another), $\mu = m\mathrm{e}$ (all goods have equal expected relative price change), and $\omega = \mathrm{e}/n$ (all goods have equal weight in the economy). Then the most efficient way of constructing k indexes from n goods is to place each of the $(k - 1)$ goods with the largest $(k - 1)$ variances in a single-good index each and to bundle the remaining $(n - k + 1)$ goods which have the smallest $(n - k + 1)$ variances into a single index.

Proof:
Substituting $\mu = m\mathrm{e}$, $\omega = (1/n)\mathrm{e}$ and $\sigma_{ij} = 0$ for all $i \neq j$ in (A.2) we get

$$A(R_{ki}) = m^2 + (1/n) \sum_{j=1}^{n} (\sigma_{jj}/n_j), \tag{A.3}$$

where n_j is the number of goods in the index in which j is included.

Without loss of generality, arrange all n goods in decreasing order of σ_{jj}. The theorem states that $A(R_{ki})$ is maximized when the first $k - 1$ goods, i.e. those with the largest values of σ_{jj}, have $n_j = 1$ and the remaining $(n - k + 1)$ goods have $n_j = n - k + 1$. The value of criterion (A.3) under this valuation rule R_k^* is

$$A(R_k^*) = m^2 + (1/n)\left\{ \sum_{j=1}^{k-1} \sigma_{jj} + (n - k + 1)^{-1} \sum_{j=k}^{n} \sigma_{jj} \right\} .$$

Suppose the value of $A(R_k^*)$ is not the maximum value and it could be improved by transferring the r^{th} good $(r \geq k)$ from the bundle of $(n - k + 1)$ goods and combining it with the t^{th} good $(t \leq k - 1)$. Under this new partition, the t^{th} good is no longer alone in a price index and the bundle of $(n - k + 1)$ goods with smaller variances is now reduced to only $(n - k)$ goods. Let us denote this valuation rule by R_k^{**}.

$$A(R_k^{**}) = m^2 + \frac{1}{n}\left\{ \sum_{j=1}^{k-1} \sigma_{jj} - \sigma_{tt} + \frac{\sigma_{tt} + \sigma_{rr}}{2} + \frac{1}{n-k} \sum_{j=k}^{n} \sigma_{jj} - \frac{\sigma_{rr}}{n-k} \right\} .$$

The change in criterion function is

$$A(R_k^{**}) - A(R_k^*) = \frac{1}{n}\left\{ \frac{-(n-k)(\sigma_{tt} - \sigma_{rr}) + 2(\sigma^* - \sigma_{rr})}{2(n-k)} \right\} ,$$

where

$$\sigma^* = \frac{1}{n - k + 1} \sum_{j=k}^{n} \sigma_{jj}.$$

The first term is always negative because $\sigma_{tt} > \sigma_{rr}$ by definition. If $\sigma_{rr} > \sigma^*$, the second term is also negative. If $\sigma^* > \sigma_{rr}$, the second term is positive but it is necessarily smaller in absolute value than the first term. Therefore, the difference $A(R_k^{**}) - A(R_k^*)$ is always negative.

Q.E.D.

Theorem 2:
Let $\Sigma = 0$ (variance of price changes is zero) and $\omega = e/n$ (all goods have equal weight in the economy). Then the most efficient way of constructing k indexes from n goods is to minimize the across-index sum of within-index sums of squared deviations of μ_i from the within-index means of μ_i.

Proof:
Substituting $\Sigma = 0$ and $\omega = (1/n)e$ in A.2,

$$\begin{aligned} A(R_{ki}) &= \sum_{u=1}^{k} \left\{ \left(\sum_{s=1}^{n_u} \sum_{t=1}^{n_u} \mu_s \mu_t \right) \Big/ (nn_u) \right\} \\ &= \sum_{u=1}^{k} \left(\sum_{s=1}^{n_u} \mu_s \right)^2 \Big/ (nn_u) \\ &= (1/n) \left\{ \sum_{j=1}^{n} \mu_j^2 - \sum_{u=1}^{k} \left(\sum_{j=1}^{n_u} \mu_j^2 - \left(\sum_{s=1}^{n_u} \mu_s \right)^2 \Big/ n_u \right) \right\}, \end{aligned}$$

where n_u is the number of goods in the u^{th} index.

The first term inside the curly brackets is a constant. The second term is the across-index sum of the within-index sums of squared deviations from the within-index means. $A(R_{ki})$ is maximized by minimizing this sum which requires that each good be grouped with others so the contribution of the good to the squared deviation from the group mean is minimized.

Q.E.D.

Theorem 3:
Let $\Sigma = \sigma^2 \mathbf{I}$ (all variances are equal and all covariances are zero), $\mu = me$ (expected changes in relative price for every commodity is equal). Then the most efficient way of constructing k indexes from n goods is to place each of the $(k - 1)$ goods with the largest weights, ω_i, in one-good indexes and to combine the remaining $(n - k + 1)$ goods into a single price index.

Proof:
Substituting $\mathbf{\Sigma} = \sigma^2\mathbf{I}$ and $\boldsymbol{\mu} = m\mathbf{e}$ into (A.2),

$$A(R_{ki}) = m^2 + \sigma^2 \sum_{j=1}^{n} (\omega_j^2/\omega_j^*),$$

where ω_j^* is the total weight of the index in which good j is included.

Without loss of generality, arrange all goods in decreasing order of ω_j. The theorem states that $A(R_{ki})$ is maximized when the first $k-1$ goods are each placed in an index by themselves and the remaining $n-k+1$ goods are bundled into a single index. Let this valuation rule be R_k^*.

$$A(R_k^*) = m^2 + \sigma^2 \left\{ \sum_{j=1}^{k-1} \omega_j + \left(\sum_{j=n-k}^{n} \omega_j^2 \right) \Big/ \left(\sum_{j=k}^{n} \omega_j \right) \right\}.$$

Suppose R_k^* is not the most accurate valuation rule and it could be improved by transferring the s^{th} good ($s \geq k$), from the bundle of $(n-k+1)$ goods and combining it with the t^{th} good, ($t \leq k-1$). Under this new arrangement, good t is no longer alone in an index by itself and the bundle of $(n-k+1)$ goods has $(n-k)$ goods in it. Let this valuation be R_k^{**}, then

$$A(R_k^{**}) = m^2 + \sigma^2 \left\{ \sum_{j=1}^{k-1} \omega_j - \omega_t + \frac{\omega_t^2 + \omega_s^2}{\omega_t + \omega_s} + \sum_{\substack{j \geq k \\ j \neq s}}^{n} \omega_j^2/(\omega^* - \omega_s) \right\},$$

where

$$\omega^* = \sum_{j \geq k} \omega_j.$$

The change in criterion function, after arranging terms, is

$$\begin{aligned}
A(R_k^{**}) - A(R_k^*) &= \sigma^2 \left\{ -\omega_t + \frac{\omega_t^2}{\omega_t + \omega_s} + \frac{\omega_s^2}{\omega_t + \omega_s} - \frac{\omega_s^2}{\omega^*} \right. \\
&\quad \left. + \left(\sum_{\substack{j=k \\ j \neq s}}^{n} \omega_j^2 \right) \left(\frac{1}{\omega^* - \omega_s} - \frac{1}{\omega^*} \right) \right\} \\
&= \sigma^2 \left\{ \frac{-\omega_t \omega_s}{\omega_t + \omega_s} + \frac{\omega_s^2(\omega^* - \omega_t - \omega_s)}{\omega^*(\omega_t + \omega_s)} \right. \\
&\quad \left. + \frac{\omega_s}{\omega^*(\omega^* - \omega_s)} \sum_{\substack{j=k \\ j \neq s}}^{n} \omega_j^2 \right\} \\
&= \frac{-2\sigma^2 \omega_s}{(\omega_t + \omega_s)(\omega^* - \omega_s)\omega^*} \left\{ \omega_s \sum_{\substack{j \geq k \\ j \neq s}}^{n} \omega_j(\omega_t - \omega_j) \right. \\
&\quad \left. + (\omega_t - \omega_s) \sum_{\substack{j \geq k \\ j \neq s}}^{n} \sum_{l > j}^{n} \omega_j \omega_l \right\}.
\end{aligned}$$

Since $(\omega_t - \omega_j)$, $j \geq k$ and $j \neq s$, as well as $(\omega_t - \omega_s)$, are always positive by construction, R_k^{**} is necessarily less accurate than R_k^*.

Q.E.D.

Theorem 4:
Let $\omega = e/n$ (all goods have equal weight in the economy), and $\mu = me$ (all goods have equal expected relative price change). Let Π_{k1} be a partition of n goods into $k > 1$ subsets. Without loss of generality let the first subset in Π_{k1} include the first n_u goods and let the second subset include the next n_v goods. Construct another k-partition of n goods, denoted by Π_{k2}, by transferring the s^{th} good, $1 \leq s \leq n_u$, from the first subset to the second subset of Π_{k1}. (Thus Π_{k1} and Π_{k2} are not comparable in fineness). Then, AMSE $(R_{k2}) <$ AMSE (R_{k1}) if and only if

$$2a - 2b + \frac{n_v + 1 - n_u}{(n_v + 1)n_u}\sigma_{ss} > \frac{c}{n_v(n_v + 1)} - \frac{d}{n_u(n_u - 1)},$$

where

$$a = \left(\sigma_{ss} + \sum_{i \in v} \sigma_{is}\right) \Big/ (n_v + 1),$$

$$b = \left(\sum_{i \in u} \sigma_{is}\right) \Big/ n_u,$$

$$c = \sum_{v} \sum_{v} \sigma_{ij},$$

$$d = \sum_{\substack{i \in u \\ i \neq s}} \sum_{\substack{j \in u \\ j \neq s}} \sigma_{ij}.$$

Proof:
Substituting $\mu = me$ and $\omega = e/n$ in (A.2), the contribution to (A.2) by index u and index v is

$$\frac{1}{nn_u} \sum_{u} \sum_{u} \sigma_{ij} + \frac{1}{nn_v} \sum_{v} \sum_{v} \sigma_{ij}.$$

The contribution to (A.2) by the indexes u' and v' created by transferring commodity s from u to v is

$$\frac{1}{n(n_u - 1)} \sum_{u'} \sum_{u'} \sigma_{ij} + \frac{1}{n(n_v + 1)} \sum_{v'} \sum_{v'} \sigma_{ij}.$$

In order to justify the transfer, the resulting index system must be more accurate than the original one; i.e. the following condition must hold:

$$\frac{1}{n(n_u - 1)} \sum_{u'} \sum_{u'} \sigma_{ij} + \frac{1}{n(n_v + 1)} \sum_{v'} \sum_{v'} \sigma_{ij} > \frac{1}{nn_u} \sum_{u} \sum_{u} \sigma_{ij} + \frac{1}{nn_v} \sum_{v} \sum_{v} \sigma_{ij}.$$

This simplfies to

$$\frac{d}{n_u - 1} + \frac{c}{n_v + 1} + \frac{\sigma_{vs}}{n_v + 1} > \frac{d}{n_u} + \frac{\sigma_{us}}{n_u} + \frac{c}{n_v},$$

where

$$c = \sum_v \sum_v \sigma_{ij},$$

$$d = \sum_{u'} \sum_{u'} \sigma_{ij},$$

$$\sigma_{vs} = 2 \sum_{i \in v} \sigma_{is} + \sigma_{ss},$$

$$\sigma_{us} = 2 \sum_{i \in u} \sigma_{is} + \sigma_{ss},$$

or

$$\frac{\sigma_{vs}}{n_v + 1} - \frac{\sigma_{us}}{n_u} > d\left(\frac{1}{n_u} - \frac{1}{n_u - 1}\right) + c\left(\frac{1}{n_v} - \frac{1}{n_v + 1}\right),$$

or

$$\frac{\sigma_{vs}}{n_v + 1} - \frac{\sigma_{us}}{n_u} > \frac{c}{n_v(n_v + 1)} - \frac{d}{n_u(n_u - 1)},$$

or

$$\frac{\sigma_{vs} + \sigma_{ss}}{n_v + 1} - \frac{\sigma_{ss}}{n_v + 1} - \frac{\sigma_{us} + \sigma_{ss}}{n_u} + \frac{\sigma_{ss}}{n_u} > \frac{c}{n_v(n_v + 1)} - \frac{d}{n_u(n_u - 1)}. \qquad \text{(A.4)}$$

By defining

$$a = [1/(n_v + 1)]\left(\sum_{i \in v} \sigma_{is} + \sigma_{ss}\right)$$

and

$$b = (1/n_u)\left(\sum_{i \in u} \sigma_{is} + \sigma_{ss}\right),$$

(A.4) can be simplified to

$$2a - 2b + \sigma_{ss}\frac{n_v + 1 - n_u}{(n_v + 1)n_u} > \frac{c}{n_v(n_v + 1)} - \frac{d}{n_u(n_u - 1)}. \qquad \text{Q.E.D.}$$

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