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Simpson's Reversal Paradox and Cost Allocation

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Allocation of indirect costs among products sometimes yields a paradoxical result that the unit cost for each product may increase under one method of allocation and decrease for each product under another method. The Stalcup Paper Company[1] case illustrates such behavior of costs and at the same time provides an accounting example of Simpson's Reversal Paradox (Simpson [1951] and Blyth [1972]) discussed in the statistics literature. As with other paradoxes, this one also disappears upon closer scrutiny. This paper examines the properties of allocated costs in order to arrive at an intuitive understanding of the results. The relationship of the cost allocation problem to Simpson's Paradox and the implications of the analysis for cost control are briefly discussed. Necessary and sufficient conditions for occurrence of the paradox are also given in Appendix A.

Cost Allocation Paradox

Table 1 shows the cost figures for a two-product department for two adjacent accounting periods.[2] There are only two types of costs: direct labor costs which have remained unchanged from period 1 to period 2 ($6.25/lb. for product *A* and $0.625/lb. for product *B*) and indirect costs whose total has also remained unchanged at $26,000. Production of *A*

* Professor, University of Minnesota and University of British Columbia. I have benefited from comments on an earlier draft of this paper from Gary Biddle, Jim Noel, Abbie Smith, and especially Yuji Ijiri, whose proof of necessary and sufficient conditions for Simpson's Paradox appears in Appendix A. [Accepted for publication June 1982.]

[1] I am grateful to Professor W. W. Cooper and R. Vancil for bringing this case to my attention.

[2] Numbers in this example are taken from the Stalcup Paper case after rounding off, changing the scale, and some simplifications to eliminate extraneous factors.

TABLE 1

A Numerical Example of Simpson's Reversal Paradox in Cost Allocation

	Year 1		Year 2	
	Product A	Product B	Product A	Product B
Quantity (lbs.)	3,200	800	2,400	2,400
Total direct labor cost ($)	20,000	500	15,000	1,500
Direct labor cost ($/lb.)	6.25	0.625	6.25	0.625
Allocation of Costs on the Basis of Direct Labor Cost				
Total indirect costs ($)	26,000		26,000	
Total direct labor costs ($)	20,500		16,500	
Burden of rate (% of DLC)	127		158	
Allocated indirect unit costs ($/lb.)	7.93	0.793	9.85	0.985
Total allocated indirect costs ($)	25,366	634	23,636	2,364
Unit cost ($/lb.)	14.18	1.418	16.10	1.61
Allocation of Costs on the Basis of Quantity				
Total indirect costs ($)	26,000		26,000	
Total quantity (lbs.)	4,000		4,800	
Allocated indirect cost ($/lb.)	6.5	6.5	5.42	5.42
Direct labor unit cost ($/lb.)	6.25	0.625	6.25	0.625
Unit cost ($/lb.)	12.75	7.125	11.67	6.045

decreased from 3,200 to 2,400 pounds and production of B increased from 800 to 2,400 pounds over the two periods. When indirect costs are allocated on the basis of direct labor dollars, unit costs increase from \$14.18 to \$16.10 for product A and from \$1.418 to \$1.61 for product B. This suggests that the efficiency of the production process, as measured by these costs, has deteriorated. Yet, when the same costs are allocated on the basis of the units (weight) of each product, the unit costs of both products decrease over the same period (from \$12.75 to \$11.67 for A and from \$7.125 to \$6.045 for B). Now it appears that the efficiency of the production process has increased.

The Stalcup Paper Company case provides an accounting example of Simpson's Reversal Paradox (Simpson [1951] and Blyth [1972]), which is stated as follows. It is possible to have:

$$P(A \mid B) > P(A \mid B') \quad (1)$$

and have at the same time both:

$$P(A \mid B \text{ and } C) \leq P(A \mid B' \text{ and } C)$$

$$P(A \mid B \text{ and } C') < P(A \mid B' \text{ and } C')$$

where $P(A \mid B)$ is the probability of event A conditional on event B and the prime indicates complements.

Consider two examples of the paradox. The first example is by Blyth [1972] in which the survival rate of patients given a standard treatment is compared to that for patients receiving a new medical treatment. These are shown in table 2.

TABLE 2
Number of Patients

Outcome	Treatment			
	Standard		New	
Died	5950	(54%)	9005	(89%)
Survived	5050	(46%)	1095	(11%)
Total	11,000	(100%)	10,100	(100%)

TABLE 3
Number of Patients

Outcome	Patient Type C				Patient Type C'			
	Treatment				Treatment			
	Standard		New		Standard		New	
Died	950	(95%)	9,000	(90%)	5,000	(50%)	5	(5%)
Survived	50	(5%)	1,000	(10%)	5,000	(50%)	95	(95%)
Total	1,000	(100%)	10,000	(100%)	10,000	(100%)	100	(100%)

Because only 11 percent of the patients given the new treatment survived, as compared to 46 percent under the standard treatment, it would seem that the new treatment is inferior to the latter. However, when we look at a set of disaggregated data for the two types of patients, as shown in table 3, this conclusion is reversed, and the new treatment appears to be definitely superior. For C-type patients the survival rate under the new treatment doubled from 5 to 10 percent and for the C'-type patients it almost doubled from 50 to 95 percent.

The reason for the reversal is that, while 10 out of every 11 patients of type C are selected for the new treatment, only 1 out of every 101 patients of type C' are given the new treatment. If event A is survival and B is the new treatment, we have the Simpson's Paradox as stated in (1):

$$P(A \mid B) = 0.11 < P(A \mid B') = 0.46$$

$$P(A \mid BC) = 0.10 > P(A \mid B'C) = 0.05$$

$$P(A \mid BC') = 0.95 > P(A \mid B'C') = 0.50.$$

Intuitively we tend to assume that $P(A \mid B)$ and $P(A \mid B')$ are equally weighted averages of $P(A \mid BC)$ and $P(A \mid BC')$ and of $P(A \mid B'C)$ and $P(A \mid B'C')$, respectively. Such an assumption is incorrect because:

$$P(A \mid B) = P(C \mid B) \cdot P(A \mid BC) + P(C' \mid B) \cdot P(A \mid BC') \quad (2)$$

$$P(A \mid B') = P(C \mid B') \cdot P(A \mid B'C) + P(C' \mid B') \cdot P(A \mid B'C').$$

In the above example, the weights $P(C \mid B)$, $P(C' \mid B)$, $P(C \mid B')$, and $P(C' \mid B')$ are 100/101, 1/11, and 10/11 by the sampling scheme. It can

TABLE 4

	Box 1	Box 2	Box 3	Box 4	Box 1 + 3	Box 2 + 4
No. of black chips	5	3	6	9	11	12
No. of white chips	6	4	3	5	9	9

easily be confirmed that:

$$0.11 = \frac{100}{101} \cdot 0.10 + \frac{1}{101} \cdot 0.95$$

$$0.46 = \frac{1}{11} \cdot 0.05 + \frac{10}{11} \cdot 0.95.$$

A second example of the paradox is provided by Gardner [1976] and shown in table 4. Given the number of black and white chips in each box, if one were to maximize the probability of drawing a black chip, box 1 is preferred over box 2 (since $5/11 > 3/7$) and box 3 is preferred over box 4 (since $6/9 > 9/14$). Yet, when the contents of boxes 1 and 3 are combined and the contents of boxes 2 and 4 are combined, the second combination is preferred over the first (since $12/21 > 11/20$). Algebraically, the paradox can be stated in terms of positive numbers. It is possible to have:

$$\frac{a+b}{c+d} > \frac{e+f}{g+h} \tag{3}$$

and at the same time have both:

$$\frac{a}{c} \leq \frac{e}{g}$$

and:

$$\frac{b}{d} \leq \frac{f}{h}.$$

The equivalence of (1) and (3) can be seen immediately by setting:

$$a = P(C|B) \cdot P(A|BC)$$
$$b = P(C'|B) \cdot P(A|BC')$$
$$c = P(C|B)$$
$$d = P(C'|B)$$
$$e = P(C|B') \cdot P(A|B'C)$$
$$f = P(C'|B') \cdot P(A|B'C')$$
$$g = P(C|B')$$
$$h = P(C'|B').$$

The cost allocation paradox mentioned above is another case of Simpson's Paradox. Let TC_i^t be the total indirect costs allocated to product i during period t on the basis of direct labor costs and q_i^t be the quantity of product i produced in period t. Then allocated unit cost is given by TC_i^t/q_i^t. If indirect costs are allocated on the basis of quantity instead of direct labor, the unit cost of period t is given by $\sum_i TC_i^t/\sum_i q_i^t$ and is the same for all i. In the example given earlier, the unit costs allocated on the basis of quantities were higher in the first year than in the second, that is:

$$\frac{TC_1^1 + TC_2^1}{q_1^1 + q_2^1} > \frac{TC_1^2 + TC_2^2}{q_1^2 + q_2^2}, \tag{4a}$$

and yet the unit costs allocated on the basis of direct labor costs were lower in the first period for each product, that is:

$$\frac{TC_1^1}{q_1^1} < \frac{TC_1^2}{q_1^2} \tag{4b}$$

and:

$$\frac{TC_2^1}{q_2^1} < \frac{TC_2^2}{q_2^2}.$$

It is easy to see that (4) is equivalent to (3).

For an intuitive understanding of the results, let us consider a simple model where a fixed indirect cost,[3] F, is allocated between two products, 1 and 2, on the basis of the amount of a given resource used directly in producing each product. It takes production of p_1 units of product 1 to use up one unit of basis resource. Similarly, production of p_2 units of product 2 requires, among other things, one unit of basis resource. In order to produce q_1 units of product 1 and q_2 units of product 2, $(q_1/p_1 + q_2/p_2)$ units of basis resource are used up.[4] Thus, total cost F is allocated between the two products as follows:

$$TC_1 = \frac{q_1 p_2}{q_1 p_2 + q_2 p_1} \cdot F. \tag{5}$$

$$TC_2 = \frac{q_2 p_1}{q_1 p_2 + q_2 p_1} \cdot F. \tag{6}$$

The corresponding per-unit (average) allocated costs are:

$$C_1 = \frac{p_2}{q_1 p_2 + q_2 p_1} \cdot F. \tag{7}$$

$$C_2 = \frac{p_1}{q_1 p_2 + q_2 p_1} \cdot F. \tag{8}$$

[3] This assumption of fixed indirect cost is relaxed later to provide more general results.

[4] If the quantity of production, measured in comparable units, is used as the basis of allocating costs, we can consider this to be an allocation on the basis of a dummy resource with $p_1 = p_2 = 1$.

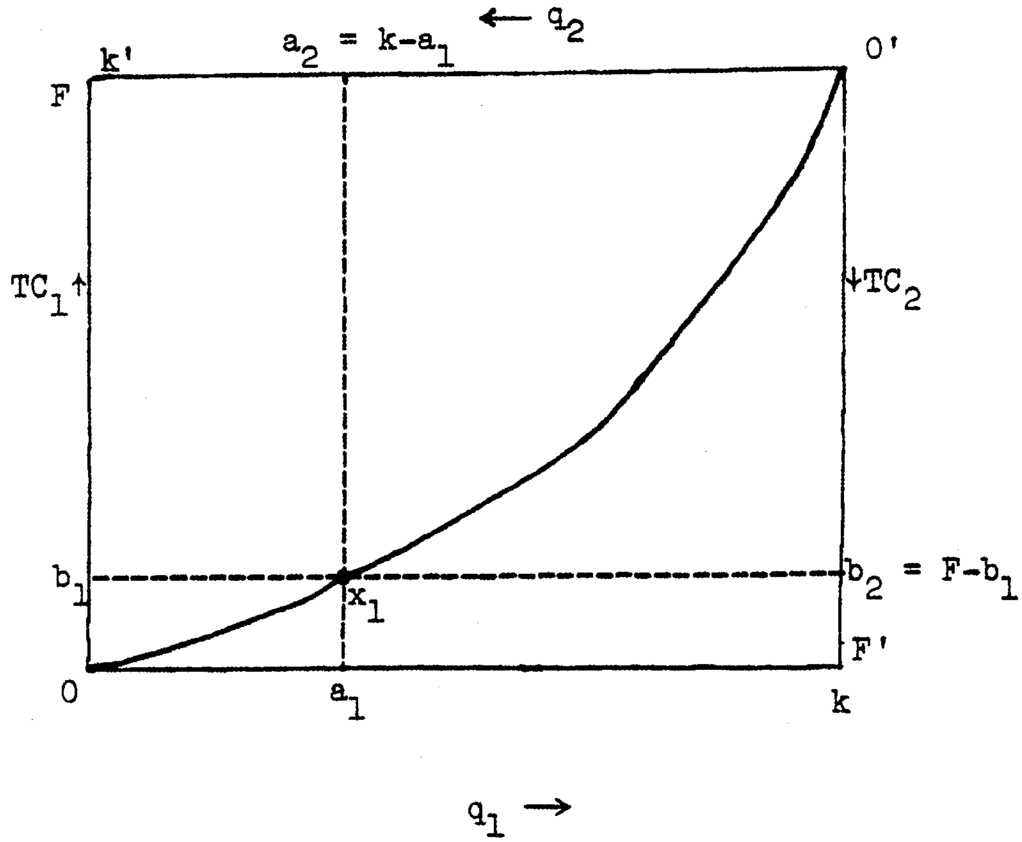

FIG. 1.—Total cost allocation curve for fixed indirect cost and fixed total production.

Consider that the behavior of total and per-unit allocated costs as the sum of q_1 and q_2 is held constant at, say, k units and the product-mix is changed by substituting a unit of one product for a unit of the other. The behavior of total costs allocated to each product, TC_1 and TC_2, when the product-mix is altered from 100 percent of product 1 to 100 percent of product 2 is shown in figure 1.

Figure 1 is a box diagram with origins for products 1 and 2 in the lower left and upper right corners, respectively. Any point x_1 on the line joining the two corners represents a product-mix of a_1 units of product 1 and $a_2 = k - a_1$ units of product 2 to which costs of b_1 and $b_2 = F - b_1$, respectively, are allocated.

The shape of the curve joining the lower left and upper right corners is determined by differentiating (5) with respect to q_1:

$$\frac{dTC_1}{dq_1} = \frac{p_1 p_2}{(q_1 p_2 + q_2 p_1)^2} \cdot F \cdot k \text{ where } k = (q_1 + q_2). \tag{9}$$

The slope of TC_1 is always positive with value $p_2 F/p_1 k$ at the lower left corner and $p_1 F/p_2 k$ at the upper right corner. If $p_1 > p_2$, the slope increases from left to right and the cost allocation curve is convex, as shown in figure 1. If $p_1 = p_2$, the slope remains constant throughout and the allocation curve is the straight line joining the two corners. Finally, if $p_1 < p_2$, the curve is concave, as shown in figure 2.

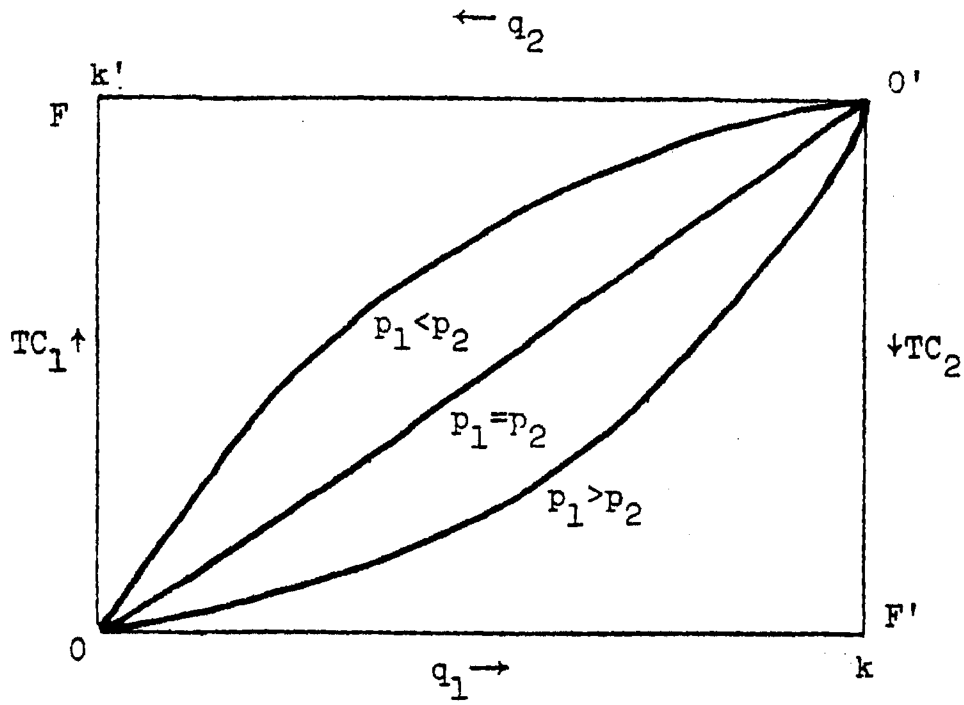

FIG. 2.—Effect of rates of direct resource utilization on cost allocation curve.

It is the curvature of the cost allocation curve, when p_1 and p_2 are not equal, that gives rise to the apparently paradoxical behavior of per-unit allocated costs. This can be seen immediately from an examination of figure 3 in which per-unit allocated costs for any product-mix are given by the slope of the line joining the point on the cost allocation curve to the appropriate origin. Take a product-mix x_1 on an allocation curve for $p_1 > p_2$ and consider a change to product-mix x_2 on the same allocation curve. The unit cost of product 1 increases from the tangent of angle x_1Ok to the tangent of angle x_2Ok. At the same time, the unit cost of product 2 increases from the tangent of angle $x_1O'k'$ to the tangent of angle $x_2O'k'$. Because of the convexity of the allocation curve throughout (increasing slope), any rightward movement in product-mix must result in increased unit costs for both products, though the total allocated costs increase for the product whose share increases. Conversely, any leftward movement must result in a decrease in the unit costs for both products, though the total allocated costs are shifted from product 1 to product 2.

Figure 4 shows similar results for cost allocation on the basis of another resource for which $p_1 < p_2$ and the cost allocation curve is concave (decreasing slope). A rightward movement in product-mix results in a decrease and a leftward movement results in an increase in the unit costs for both products.

The behavior of marginal and average (per-unit) allocated costs as a function of the product-mix is shown in figures 5 and 6. In figure 5, for $p_1 > p_2$, average costs for both products increase continuously (from p_2F/p_1k to F/k for product 1 and from F/k to p_1F/p_2k for product 2) as q_1 increases from zero to k (and q_2 decreases from k to zero). The marginal

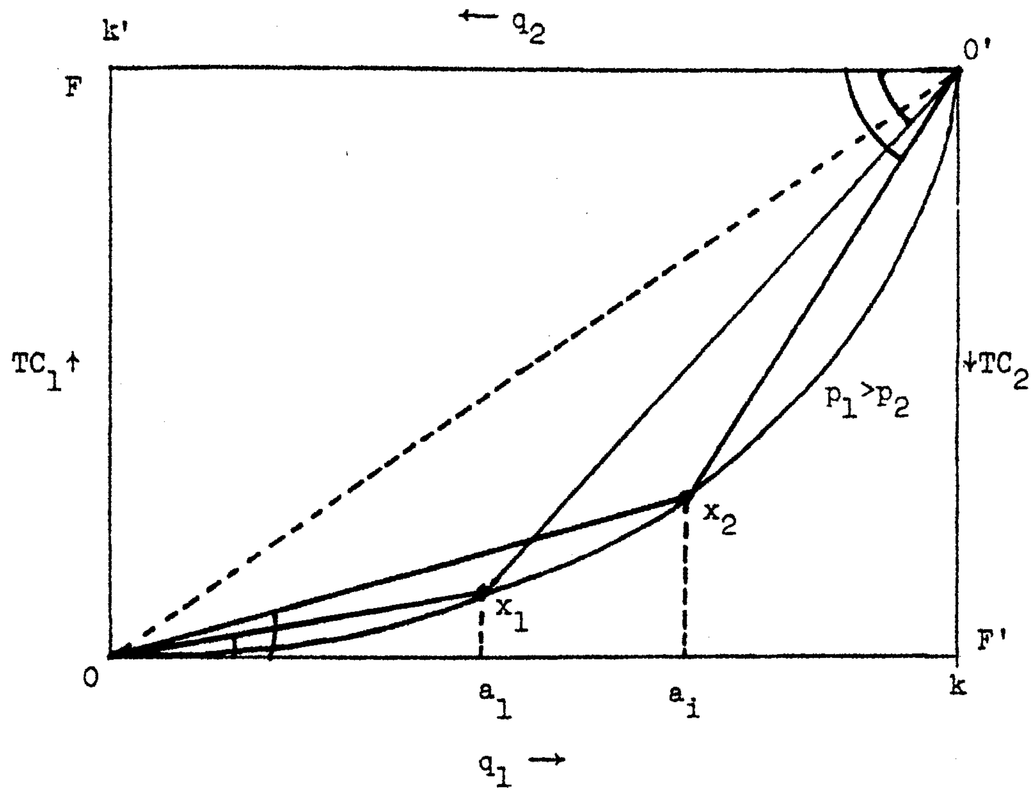

FIG. 3.—Effect of movement along the cost allocation curve on per-unit allocated costs.

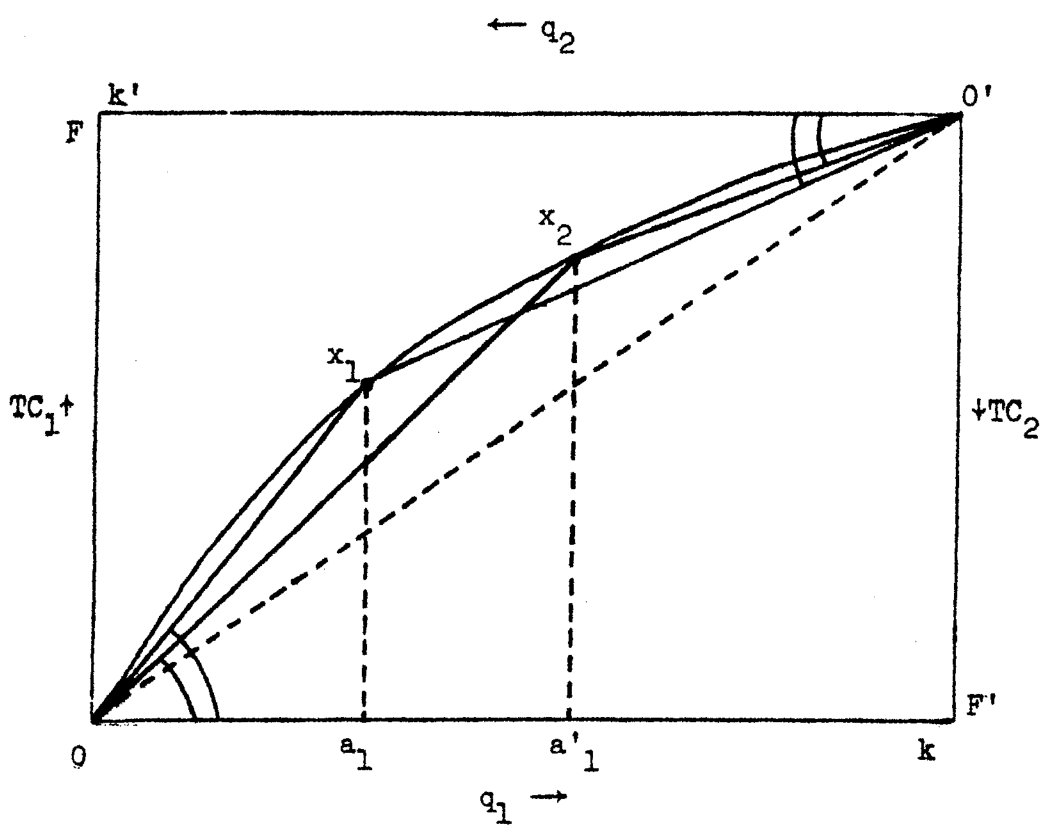

FIG. 4.—Effect of movement along the cost allocation curve on per-unit allocated costs.

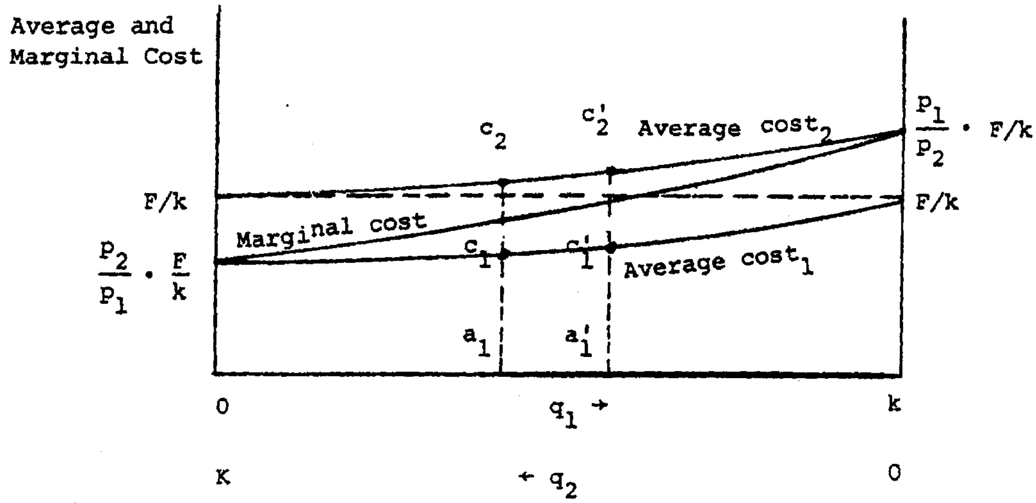

FIG. 5.—Behavior of average and marginal allocated costs: for $p_1 > p_2$.

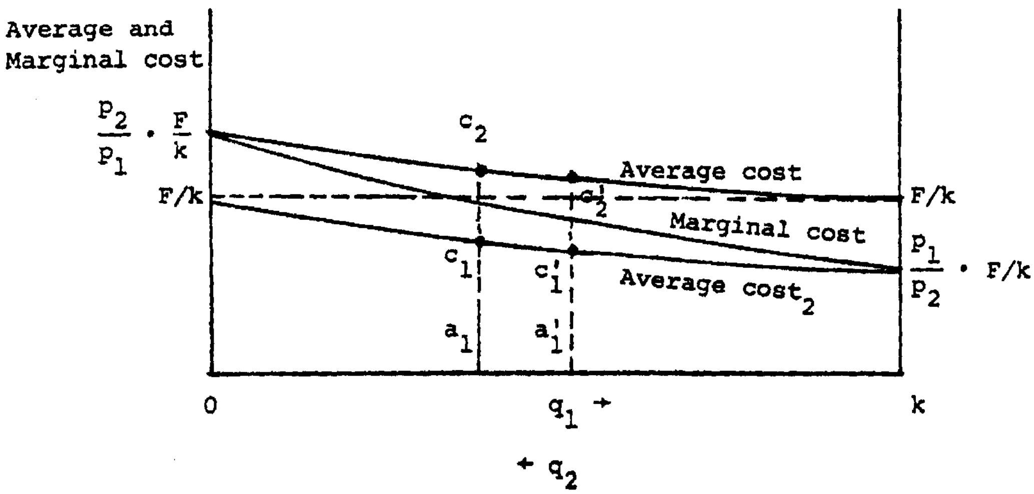

FIG. 6.—Behavior of average and marginal allocated costs: for $p_1 < p_2$.

cost is positive and increasing from p_2F/p_1k to p_1F/p_2k over the same ranges giving convexity of total allocated costs in figure 1.

When quantity a_1 of product 1 is produced, the per-unit costs for products 1 and 2 are given by ordinate c_1 and c_2 respectively in figure 5 (c_1 and c_2 are equal to the tangents of $x_1 0k$ and $x_1 0'k'$, respectively, in figure 3). The weighted average of these unit costs, with weights a_1 for c_1 and $a_2 = k - a_1$ for c_2, is easily seen to be equal to F/k from equations (7) and (8), and this weighted average is given by the dotted horizontal line. When the product-mix is changed by increasing the quantity of product 1 from a_1 to a'_1, both average costs increase (from c_1 to c'_1 and from c_2 to c'_2) but their weighted average remains unchanged at F/k because the relative weights have shifted in favor of product 1 which has the lower average cost.

Similarly, in figure 6, when $p_1 < p_2$ the per-unit costs for each product-mix decline with a shift in product-mix in favor of product 1 but the weighted average cost remains unchanged at F/k. Thus the choice of allocation basis determines whether the unit costs increase or decrease. The sign of change is the same for both products, and this sign cannot be used to draw inferences about the efficiency of the production process itself.

In figures 5 and 6, marginal and average cost curves are convex and approach the horizontal dotted line at F/k as the ratio p_1/p_2 approaches unity. This implies that the apparent paradox of allocated costs will appear only when the number of units of basis resource used in making one unit each of various products is unequal. More formally, Ijiri has worked out the necessary and sufficient conditions for occurrence of the Paradox which appear in Appendix A.

Generalization of Results

For the sake of simplicity, the above results are given for fixed indirect costs and constant sum of the units of two products. When the indirect costs, I, and the outputs of the two products are allowed to vary in any manner, the sign of change in the unit costs for both products due to a change in output is still the same. The sign of change in unit costs due to a small increase in output q_1 depends on whether:

$$\frac{p_1 \dfrac{dq_2}{dq_1} + p_2}{q_1 p_2 + q_2 p_1} \quad \text{is less than or greater than} \quad \frac{dI(\cdot)}{dq_1}\bigg/ I(\cdot). \tag{10}$$

In the simplified example given above, $\dfrac{dq_2}{dq_1} = -1$ and $\dfrac{dI(\cdot)}{dq_1} = 0$; thus criterion (10) takes the form of $(p_2 - p_1) < \text{or} > 0$. Note that this condition can be used to determine the sign of change in unit costs caused by any incremental change in the quantity of product 1. Further, the condition can be used to determine the sign of change in unit costs allocated on the basis of any resource used in production (as long as p_1 and p_2 are known) irrespective of whether indirect cost $I(\cdot)$ is dependent on that resource. Of course, the manager may not find the cost allocations based on resources which do not appear in function $I(\cdot)$ useful. If costs are allocated on the basis of quantity of product itself measured in identical units, we can use $p_1 = p_2 = 1$ for a dummy resource, in which case the apparent paradox of unit costs disappears.

Although the three examples of Simpson's Paradox given above (medical, accounting, and chips in the box) reflect the same phenomenon, the implications of the reversal for decisions based on data are quite different. For the medical example, the new treatment is truly better than the standard (assuming that the sampling frequencies are sufficiently close to the underlying population distributions) and the decision to select a treatment on the basis of disaggregated data is better than the decision on the basis of aggregated data. For the chips in the box example, the objective of maximizing the probability of drawing a black chip is maximized by selecting box 1 over 2, 3 over 4, and (2 + 4) or (1 + 3). Different levels of aggregation result in different decisions, and, unlike the medical example, there is no single right decision. In the accounting example, neither the aggregated nor the disaggregated data tell us much on which we can base our decision to reward or admonish the production manager.

Implications for Cost Control

Paradoxes are more than just logical curiosities since they play upon the darker regions of intuition and can, perhaps, be used to sharpen our intuition through derivation of new rules of thumb. In earlier sections we saw that, as long as the rate of utilization of basis resource (p_1 and p_2) remains unchanged, a change in per-unit allocated costs brought about by any change in output, product-mix, or indirect costs must have the same sign for all products.[5] The inverse of this proposition is important for managers concerned with cost control.

PROPOSITION: If the sign of change in per-unit allocated costs for all products is not the same, the units of basis-resource required to produce each unit of one or more products must have changed.

In other words, similarity in the sign of change in allocated per-unit costs is the norm, and whenever the sign is not the same, it provides the manager with a valuable clue to the changing cost-quantity relationships that may need further investigation. Not all such changes will be revealed by the difference in sign of changes in unit costs; the necessary and sufficient condition for opposite signs of changes in unit costs is:

$$N_2^2/N_2^1 \gtreqless I^2/I^1 \gtreqless N_1^2/N_1^1$$

where

N_i^t = units of product i that could be produced from basis resource actually used in period t; and

I^t = total costs to be allocated between products in period t.

In the Stalcup Paper case, the rate of utilization of basis resource, direct labor, has remained unchanged over the two years and therefore the sign of change in per-unit allocated costs for two products will always be identical, irrespective of what the total indirect costs and the quantities produced are. Because the choice of allocation basis and the rate of utilization of basis resource have such major effects on per-unit allocated costs of products, it is again useful to issue another call for vigilance in interpreting allocated cost data for different products. Nevertheless, allocated per-unit costs can still be used as a means of monitoring the rate of utilization of the resource used as the basis of cost allocation.

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[5] The proposition can be checked by comparing the signs of the partial derivatives of c_1 and c_2 (in equations (7) and (8)) with respect to parameters p_1, p_2, q_1, q_2, F, and $k = q_1 + q_2$. Partial derivatives with respect to p_1 and p_2 are the only ones that have different signs for c_1 and c_2.

APPENDIX A

Ijiri's Necessary and Sufficient Conditions for Simpson's Paradox

Let $0 \leq p, q, r, s \geq 1$; $p > r$; $q > s$; and, without loss of generality, $p \geq q$. Then, Simpson's Paradox occurs if:

$$up + (1 - u)q < vr + (1 - v)s, \tag{A1}$$

where $0 < u, v < 1$. The necessary and sufficient condition for the existence of u and v such that (A1) holds is:

$$r > q. \tag{A2}$$

Proof

(Necessity): $q \leq up + (1 - u)q$ by definition (equality iff $p = q$). Similarly, $vr + (1 - v)s \leq \max(r, s)$ by definition (equality iff $r = s$). Hence, from (A1), $q < \max(r, s)$. But $q > s$. Hence $q < r$.

(Sufficiency): if (A2) holds, select t such that $r > t > q$ and let $u' = (t - q)/(p - q)$ and $v' = (t - s)/(r - s)$. Then (A1) is satisfied for

(*a*) $u = u'$ and any $v > v'$: because
$up + (1 - u)q = u'(p - q) + q = t$ while
$vr + (1 - v)s = v(r - s) + s > v'(r - s) + s = t$,

(*b*) $v = v'$ and any $u < u'$: because
$vr + (1 - v)s = v'(r - s) + s = t$ while
$up + (1 - u)q = u(p - q) + q < u'(p - q) + q = t$.

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