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Stationarity of Market Risk: Random Coefficients Tests for Individual Stocks

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THE PURPOSE OF THIS paper is threefold: First, a procedure for obtaining unbiased and consistent estimates of the variance of the disturbance term (step variance) in the market risk of individual stocks and portfolios is presented under the random walk and autoregressive hypotheses for the market risk. These estimates can be used to test the null hypothesis that the market risk of a given stock over a given time series is stationary against the abovementioned alternative hypothesis about the nature of nonstationarity in the market risk of stock. Second, under the assumption that the market risk follows a random walk, estimates of step variance of the market risk of the stocks listed on the New York Stock Exchange are presented. The null hypothesis of stationary risk is tested against the random walk hypothesis for each stock over the period 1926–1975 and its subintervals. In the third part, the effect of portfolio diversification on nonstationarity of the market risk of portfolios is examined.

A considerable amount of attention has been paid to the tests of hypotheses about the nonstationarity of market risk of common stocks. The methodology used involved cross-sectional correlation and other forms of analysis of the OLS regression estimates of risk of individual stocks over two or more contiguous segments of time series. Examples of this work include Blume [3], Levy [12], Sharpe and Cooper [15], Fisher [6, 7], and Fisher and Kamin [8]. Blume concluded that the market risk estimated over one period persists into future periods but did not look stationary. After detailed testing, Bogue [4] reached a similar conclusion. Fisher and Kamin [8] have conducted tests of the stationarity hypothesis and concluded that the behavior of market risk is best described by a random walk or a first-order autoregressive process with a serial correlation very close to one:

$$\beta_t = \rho\beta_{t-1} + (1-\rho)\beta + v_t \qquad (1)$$

where $E(v_t) = 0$, $E(v_t^2) = \sigma_v^2$, $E(v_t v_s) = 0$, $t \neq s$. In autoregressive model (1) of market risk, ρ is the first-order serial correlation coefficient and σ_v^2 is the variance of the disturbance term in the market risk process. The variance of β_t is given by $\sigma_v^2/(1-\rho^2)$, which is equal to σ_v^2 for a mean reverting process ($\rho = 0$) and approaches infinity as the process approaches random walk ($\rho \to 1$). We shall refer to σ_v^2 as the step variance of the market risk process. Fisher and Kamin [8] have presented the estimates of K, the ratio of σ_v^2 to the residual variance of the

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market model, under the assumption that this parameter is the same for all stocks.

The approach used in the present study to estimate step variance (σ_v^2) and to test the stationarity hypothesis differs from the previous studies in an important respect: the estimation and testing are carried out for individual stocks and portfolios. Optimal estimation of market risk of stocks from return data involves the use of weighted regression with the distribution of weights determined by the extent of nonstationarity of market risk.[1] The procedure presented here provides unbiased and consistent estimates of σ_v^2 for individual stocks for various intervals of time in order to make it possible to obtain optimal estimates of market risk in the presence of nonstationarity.

In order to obtain unbiased and consistent estimates of the step variance of market risk, the market model is reformulated as a random coefficients model which is transformed into a simple linear model whose coefficients are slope and error variances of the market model. This approach obviates the need for obtaining OLS estimates of β before drawing inferences about the stationarity of β over time. The random coefficients model provides a more powerful test of the stationarity hypothesis than the OLS procedures used by Blume [2] and others because the hypothesis is tested parametrically against a specific alternative hypothesis of autoregressive process.

While the hypothesis about the nonstationarity of market risk of stocks in general has been supported by previous studies, results for the behavior of market risk of individual stocks and groups of stocks over various parts of the available time series data are presented here for the first time. It has often been argued that even in the presence of nonstationarity in the market risk of common stocks, the risk of nonmanaged portfolios is reasonably stationary over time and, therefore, nonstationarity of individual stocks is not significant from the point of view of portfolio management. The estimates of variance of the market risk of portfolios and tests of significance on such estimates indicate that when the risk of individual stocks is nonstationary, diversification does not diminish the statistical significance of nonstationarity in spite of a decrease in the step variance of portfolio risk.

Random Coefficients Model

The market model for a given stock can be written as

$$y_t = \alpha + \beta_t x_t + \epsilon_t, \tag{2}$$

where subscript t on β indicates that the risk of the stock is subject to change from time to time. First, consider the simple case where β_t is an independently distributed random variable with mean β and variance σ_β^2. β_t is also independent of x_t and ϵ_t. Model (2) can be rewritten as

$$y_t = \alpha + \beta\, x_t + u_t \tag{3}$$

$$u_t = (\beta_t - \beta)x_t + \epsilon_t. \tag{4}$$

[1] Cooley and Prescott [5], Rosenberg [13], Rosenberg and McKibben [14], Fisher [6, 7], Bogue [4], and Sunder [17]

The conditional variance of u_t is

$$\mathrm{Var}(u_t|x_t) = \sigma_\epsilon^2 + \sigma_\beta^2 x_t^2. \tag{5}$$

Equation (5) can be viewed as a linear model with dependent variable $\mathrm{Var}(u_t|x_t)$, independent variable x_t^2, intercept σ_ϵ^2 and slope σ_β^2.[2] $\mathrm{Var}(u_t|x_t)$, however, is not directly observable. Let $\hat{u}_t$ be the estimate of u_t obtained from the simple ordinary least square regression y_t on x_t shown in (3). Then, it can be shown that

$$\mathrm{Var}(\hat{u}_t) = A_{1t}\sigma_\epsilon^2 + A_{2t}\sigma_\beta^2 \tag{6}$$

where

$$A_{1t} = 1 - \frac{1}{n} - \frac{(x_t - \bar{x})^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \tag{7}$$

$$A_{2t} = x_t^2\left(1 - \frac{2}{n} - \frac{2(x_t - \bar{x})^2}{\sum_{i=1}^n (x_i - \bar{x})^2}\right) + \frac{\sum_{i=1}^n x_i^2}{n^2}$$

$$+ \frac{(x_t - \bar{x})^2}{(\sum_{i=1}^n (x_i - \bar{x})^2)^2} \sum_{i=1}^n x_i^2(x_i - \bar{x})^2 + \frac{2(x_t - \bar{x})}{n\sum_{i=1}^n (x_i - \bar{x})^2} \sum_{i=1}^n x_i^2(x_i - \bar{x}). \tag{8}$$

Since $E(\hat{u}_t) = 0$, $E(\hat{u}_t^2) = \mathrm{Var}(\hat{u}_t)$ and we can write

$$\hat{u}_t^2 = \mathrm{Var}(\hat{u}_t) + w_t, \tag{9}$$

where $E(w_t) = 0$. Substituting the value of $\mathrm{Var}(\hat{u}_t)$ in (9), we have

$$\hat{u}_t^2 = \sigma_\epsilon^2 A_{1t} + \sigma_\beta^2 A_{2t} + w_t. \tag{10}$$

Theil and Mennes [18] suggested that equation (10) has the form of a linear model, and the ordinary least square regression of $\hat{u}_t^2$ on A_{1t} and A_{2t}, all of which are functions of observable variables y_t and x_t, ($t = 1, 2, \cdots n$), yields unbiased though inefficient estimates of coefficients σ_ϵ^2 and σ_β^2. Hildreth and Houck [10] generalized Theil and Mennes' results to regressions involving more than two random coefficients and showed that under reasonable assumptions,[3] variance of OLS estimators of σ_ϵ^2 and σ_β^2 obtained from (10) is of the order of $(1/n)$ (where n is the number of observations) and therefore these estimators are consistent. We

[2] From (5), note that the nonstationarity of β ($\sigma_\beta^2 > 0$) implies that the OLS residuals u_t is heteroscedastic in x_t even if the true disturbance term ϵ_t in the market model is homoscedastic. Thus, evidence presented by Fisher and Kamin [8, Figure 1] is consistent with nonstationarity of β and does not necessarily imply that ϵ_t is heteroscedastic.

[3] The assumptions are:

(i) $a_{11} = \underset{n\to\infty}{Lt} \frac{1}{n}\sum_{t=1}^n A_{1t}^2$, $a_{22} = \underset{n\to\infty}{Lt} \frac{1}{n}\sum_{t=1}^n A_{2t}^2$, and $a_{12} = \underset{n\to\infty}{Lt} \frac{1}{n}\sum_{t=1}^n A_{1t}A_{2t}$ are finite;

(ii) Matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ is nonsingular;

(iii) Random variables ϵ_t and $(\beta_t - \beta)$ have finite fourth moments, and

(iv) Sequence $\{x_t\}$ is bounded.

shall use $\hat{\sigma}_\epsilon^2$ and $\hat{\sigma}_\beta^2$ to represent the OLS estimators of σ_ϵ^2 and σ_β^2 respectively from (10).

Since the covariance matrix of w_t is not known, OLS estimators $\hat{\sigma}_\epsilon^2$ and $\hat{\sigma}_\beta^2$ do not satisfy the conditions of the Gauss-Markov theorem and are not the minimum variance estimators. Theil and Mennes have shown that the covariance matrix of w_t itself is a function of unknown parameters σ_ϵ^2 and σ_β^2. Under simplifying assumptions,[4] Theil and Mennes suggest the use of estimates $\hat{\sigma}_\epsilon^2$ and $\hat{\sigma}_\beta^2$ to approximate the covariance matrix of w_t, whose off-diagonal terms are shown to be close to zero. This estimated covariance matrix can be used to obtain generalized least square estimates, σ_ϵ^{*2} and σ_β^{*2} from (10). These estimates can be used to obtain the covariance matrix of u_t in (3) and generalized least square estimates of α and β.[5]

Relaxing the Independence Assumption

The estimation procedures given above apply to the case when the distribution of β_t is intertemporally independent. From both theoretical reasons[6] as well as empirical evidence provided by the previous studies,[7] this assumption is not accurate. The market risk of stock is more likely to follow a random walk or autoregressive process. If random walk is the alternative hypothesis against which the null hypothesis of constant β is to be tested, we are interested in estimation of the step variance of the random walk process, i.e., the variance σ_v^2 of the change in β from $(t-1)$ to t.

To account for serial dependence in β_t, regression model (6) is modified in such a way that coefficient σ_β^2 of A_{2t} is replaced by the variance, σ_v^2, of the disturbance term in the autoregressive process (1) that describes the time series behavior of β_t. Model (3) can be written in matrix notation:

$$\underline{y} = X\underline{\gamma} + \underline{u}.$$

Let $M = (I - X(X'X)^{-1}X')$ and denote the ij^{th} element of M by m_{ij}. Then equation (10) can be rewritten as

$$\hat{u}_t^2 = \sigma_\epsilon^2 m_{tt} + \sigma_\beta^2 \cdot \sum_{i=1}^{n} m_{ti}^2 x_i^2 + w_t. \tag{12}$$

[4] The assumptions are:

(i) Mean value of independent variable x_t is zero;

(ii) The number of observations, n, is very large.

[5] As estimators of a nonnegative parameter, both $\hat{\sigma}_\beta^2$ and σ_β^{*2} have a serious flaw since they are not restricted from being negative. Nonnegative estimators of σ_β^2 such as

$$\tilde{\sigma}_\beta^2 = \max(\hat{\sigma}_\beta^2, 0) \tag{11}$$

can be easily constructed. Estimator $\tilde{\sigma}_\beta^2$ clearly has a lower mean square error than $\hat{\sigma}_\beta^2$ but it also has a positive bias which is a serious defect from the point of view of hypothesis testing. We shall, therefore, use unbiased estimators of σ_β^2 which are not restricted from being negative.

[6] Each firm consists of a portfolio of assets that gradually changes over time as old assets are discarded and new ones are added. When the risk of the discarded or wasted assets is not equal to the risk of newly acquired assets, the risk of the entire portfolio of assets, i.e., the firm, also changes over time. The longer the elapsed time interval, the larger is the change in risk likely to be. Since firms rarely change all their assets at once, their risk in adjacent periods is likely to be similar.

[7] See Fisher and Kamin [8], and Bogue [4].

If a random walk process ($\rho = 1$ in (1)) is the alternative hypothesis, (12) is modified to[8]

$$\hat{u}_t^2 = \sigma_\epsilon^2 m_{tt} + \sigma_v^2 \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ti} m_{tj} x_i x_j \min(i, j) + w_t. \tag{13}$$

Unbiased estimates of σ_ϵ^2, the intercept variance, and σ_v^2, the step variance of the slope, can be obtained by ordinary least squares to estimate (13).

In the more general case when β_t follows an autoregressive process (1) with step variance σ_v^2 and serial correlation ρ, (12) is further modified to:

$$\hat{u}_t^2 = \sigma_\epsilon^2 m_{tt} + \sigma_v^2 \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ti} m_{tj} x_i x_j \phi_{ij} + w_t \tag{14}$$

where $\phi_{ij} = \dfrac{\rho^{i+j}}{1-\rho^2} + \rho^{|i-j|} \sum_{k=0}^{l-1} \rho^{2k}$ and $l = \min(i, j)$,

and unbiased estimates of σ_ϵ^2 and σ_v^2 can be obtained from OLS regression on (14).[9]

Tests of Significance

Under well-known conditions,[10] the ratio of OLS estimates of linear coefficients to their standard error of estimation have a Student-t distribution, and this property is often utilized in conducting tests of significance on linear coefficients. Since the distribution of w_t is not normal, these conditions are clearly not fulfilled by the linear model (10), and therefore "t-tests" on coefficients of this model provide only an approximation. Since coefficient estimates $\hat{\sigma}_v^2$ and $\hat{\sigma}_\epsilon^2$ are unbiased, under the null hypothesis $\sigma_v^2 = 0$, the "t-ratio" has zero mean. Monte Carlo experiments ($n = 50$, 100 replications) indicated that the relative frequency of observing a "t-ratio" larger than 1.64 (which corresponds to the right-tailed test at the 5 percent level of significance for a normal distribution or t-distribution with very large degrees of freedom) is about 6 percent. Therefore, if we reject the null hypothesis $\sigma_v^2 = 0$ whenever the ratio of $\hat{\sigma}_v^2$ to its standard error is greater than 1.64, we commit a type I error only about 6 percent of the time.[11] Monte Carlo experiments also indicated that (1) both mean and median of a "t-ratio" are slightly negative, (2) distribution of a "t-ratio" is skewed to the right and (3) the variance of a "t-ratio" is close to 1 under the null hypothesis but increases substantially when σ_v^2 is increased.

Results

Summary statistics for estimates of step variance of the market risk ($\hat{\sigma}_v^2$) of individual NYSE stocks under the random walk hypothesis are given in Table

[8] Derivation of equations (13) and (14) is available from author upon request.

[9] Fisher and Kamin's [8] K is the ratio of their estimates for individual stocks.

[10] See Graybill [9, Ch. 6].

[11] Power functions and other details of the test obtained from Monte Carlo experiments are given in Sunder [16, pp. 114-126].

Table I

Tests of Stability of Relative Risk of Common Stocks

Data and Test Interval	Number of Firms		Cross-Sectional Summary Statistics: Average	Standard Deviation	Minimum	Fractiles: 10%	20%	30%	40%	50%	60%	70%	80%	90%	Maximum	Per-Cent > 0	Per-Cent > 1.64
Jan. 1926–Dec. 1975 (600 months)	127	$\hat{\sigma}^2_\epsilon$	.0019	.0016	.0002	.0006	.0007	.0009	.0011	.0014	.0018	.0020	.0026	.0037	.0091		
		t-value	4.70	2.87	.24	1.50	2.39	3.05	3.83	4.33	4.91	5.46	6.33	8.05	17.75	100	88
Jan. 1926–Dec. 1950 (300 months)	262	$\hat{\sigma}^2_\epsilon$	.0069	.0077	−.0047	.0016	.0021	.0029	.0037	.0049	.0058	.0075	.0098	.0144	.0633		
		t-value	3.34	2.04	−.59	.78	1.37	2.10	2.57	3.26	3.72	4.32	5.07	5.90	11.62	96	77
Jan. 1951–Dec. 1975 (300 months)	493	$\hat{\sigma}^2_\epsilon$	.0034	.0057	−.0066	−.0009	−.0000	.0005	.0010	.0019	.0027	.0041	.0056	.0096	.0478		
		t-value	1.82	2.61	−1.34	−.49	−.02	.31	.69	1.09	1.57	2.24	3.21	4.84	17.43	79	39
Jan 1926–June 1938 (150 months)	308	$\hat{\sigma}^2_\epsilon$	.0049	.0233	−.0282	−.0021	−.0002	.0006	.0012	.0021	.0031	.0045	.0064	.0104	.3927		
		t-value	1.03	1.58	−1.37	−.36	−.06	.20	.43	.70	1.05	1.40	1.71	2.57	13.40	77	22
July 1938–Dec. 1950 (150 months)	684	$\hat{\sigma}^2_\epsilon$	.0113	.0259	−.0454	−.0014	.0006	.0021	.0038	.0054	.0077	.0110	.0159	.0252	.3121		
		t-value	2.46	2.74	−1.61	−.33	.17	.65	1.20	1.77	2.31	3.30	4.37	6.43	14.32	83	53
Jan. 1951–June 1963 (150 months)	783	$\hat{\sigma}^2_\epsilon$	.0015	.0175	−.0660	−.0087	−.0049	−.0032	−.0015	−.0001	.0013	.0034	.0066	.0121	.3605		
		t-value	.20	1.18	−1.87	−.99	−.72	−.50	−.25	−.02	.24	.54	1.00	1.56	7.35	51	8
July 1963–Dec. 1975 (150 months)	759	$\hat{\sigma}^2_\epsilon$	.0056	.0129	−.0185	−.0037	−.0017	−.0003	.0005	.0021	.0039	.0066	.0105	.0185	.1402		
		t-value	1.11	2.15	−1.63	−.65	−.36	−.08	.14	.41	.80	1.25	2.26	3.65	12.85	66	26

Jan. 1926–Mar. 1932 (75 months)	356	$\hat{\sigma}_*^2$	.0166	.0495	−.1067	−.0170	−.0056	−.0009	.0023	.0072	.0133	.0210	.0301	.0576	.5111		
		t-value	.78	1.43	−1.83	−.67	−.35	−.05	.14	.43	.77	1.15	1.74	2.70	7.50	67	28
Apr. 1932–June 1938 (75 months)	596	$\hat{\sigma}_*^2$	.0048	.0202	−.0227	−.0048	−.0021	−.0003	.0009	.0019	.0036	.0054	.0077	.0138	.4158		
		t-value	.66	1.18	−1.60	−.59	−.33	−.09	.23	.54	.81	1.13	1.48	1.97	7.04	67	17
July 1938–Sept. 1944 (75 months)	711	$\hat{\sigma}_*^2$	.0264	.0535	−.0994	−.0026	.0020	.0056	.0090	.0135	.0174	.0244	.0372	.0616	.5788		
		t-value	1.81	1.80	−1.33	−.30	.21	.65	1.08	1.49	1.96	2.58	3.41	4.52	7.53	84	47
Oct. 1944–Dec. 1950 (75 months)	788	$\hat{\sigma}_*^2$	.0047	.0374	−.1487	−.0228	−.0128	−.0070	−.0034	.0008	.0044	.0099	.0167	.0344	.3550		
		t-value	.17	1.05	−2.36	−1.04	−.70	−.45	−.20	.04	.27	.63	1.00	1.57	4.50	52	8
Jan. 1951–Mar. 1957 (75 months)	928	$\hat{\sigma}_*^2$	.0102	.1149	−.3422	−.0353	−.0179	−.0100	−.0043	.0013	.0072	.0168	.0290	.0557	3.0906		
		t-value	.26	1.23	−2.20	−1.09	−.70	−.47	−.21	.07	.33	.69	1.10	1.84	6.16	52	13
Apr. 1957–June 1963 (75 months)	898	$\hat{\sigma}_*^2$	−.0020	.0334	−.2188	−.0275	−.173	−.0111	−.0069	−.0036	−.0006	.0030	.0095	.0221	.3538		
		t-value	−.08	1.10	−2.38	−1.22	−.92	−.70	−.49	−.28	−.05	.23	.59	1.30	5.03	48	3
July 1963–Sept. 1969 (75 months)	875	$\hat{\sigma}_*^2$	−.0114	.0441	−.2651	−.0523	−.0320	−.0221	−.0152	−.0097	−.0041	.0015	.0111	.0264	.2572		
		t-value	−.30	.98	−2.45	−1.38	−1.13	−.88	−.63	−.44	−.18	.07	.46	1.00	4.13	32	2
Oct. 1969–Dec. 1975 (75 months)	1106	$\hat{\sigma}_*^2$	.0064	.0251	−.0693	−.0123	−.0072	−.0043	−.0016	.0011	.0043	.0086	.0164	.0284	.2370		
		t-value	.51	1.50	−1.72	−.82	−.58	−.37	−.15	.12	.42	.82	1.35	2.34	9.04	54	17

I.[12] At the top of Table I are the statistics for estimates of step variance over the 600-month period from January, 1926 to December, 1975. One hundred and twenty-seven stocks satisfied the minimum data requirement that a complete record of return data be available for the 600 months. A similar minimum data requirement has been used for results presented for other intervals. We shall return later to the probable effects of the minimum data requirement on the results presented here.

The mean and median of estimated variance of the monthly changes in the market risk of 127 stocks during the fifty-year period 1926-75 are 0.0019 and 0.0014, respectively. The mean value corresponds to a standard deviation of 0.0436 for monthly and 0.151 for yearly changes in the market risk of a stock. This magnitude of nonstationarity of risk appears to be nontrivial. On the basis of this average, the difference between the risk of a stock in two consecutive years would be greater than 0.15 in magnitude for one out of every three cases. An error of 0.15 in estimation of market risk implies an error of 1 percent in estimated expected return on the stock under the Sharpe-Lintner model if the expected risk premium is assumed to be 6.7 percent.

Every one of the 127 individual stocks has a positive estimate of $\hat{\sigma}_v^2$, while under the null hypothesis, less than 50 percent would be expected to have positive values of $\hat{\sigma}_v^2$. Under the null hypothesis, only about 6 percent of the "t-ratios" would be expected to exceed 1.64. About 88 percent of the stocks have "t-ratios" greater than 1.64. This evidence soundly rejects the hypothesis that the risk of individual stocks over this fifty-year period was stationary ($\sigma_v^2 = 0$).

The remaining parts of Table I present the summary statistics for estimates of step variance σ_v^2 of individual stocks for 300, 150, and 75-month segments of the time series. Power of the test is reduced with the reduction in the number of observations. All stocks for which a complete return record over the respective interval is available on the CRSP file have been included. Evidence of nonstationarity in the twenty-five year period 1926-50 is almost as strong as in the entire fifty-year period with cross-sectional mean and median values of $\hat{\sigma}_v^2$ equal to 0.0069 and 0.0049, respectively. Over 96 percent of the "t-ratios" are positive and over 77 percent exceed the critical level of 1.64. Since both the Depression and the World War II years are included in this time series, strong evidence on nonstationarity of risk of individual stocks is not surprising. Evidence of nonstationarity of risk during the latter half of the time series, 1951-75, is still strong, though the level of nonstationarity is lower in this period. Mean and median values of $\hat{\sigma}_v^2$ are 0.0034 and 0.0019, respectively, with 79 percent of the estimates greater than zero. Over 39 percent of the t-values are greater than 1.64.

The next four rows of Table I show the results of estimation for four 150-month segments of the time series. The null hypothesis of stationary risk is rejected at the 6 percent level for about 23 percent, 53 percent, 8 percent and 26 percent of the stocks in 1926-38, 1938-50, 1951-63, and 1963-75, respectively. As one might expect, the accuracy of estimates suffers as fewer observations are used. During

[12] June 1976 version of the CRSP Monthly Return file at the University of Chicago was used for these results.

1951-63, the market risk of individual stocks was just about stationary. In all other 150 month intervals, there has been considerable nonstationarity.

The last segment of Table I shows results for eight 75-month segments of the time series. The significance of this evidence of nonstationarity weakens as the number of observations diminishes and the test interval is shortened. In five 75-month segments beginning October 1944, there is little evidence of nonstationarity. In the first three 75-month segments, significant evidence of nonstationarity can be seen.[13]

The estimates of variance of changes in the market risk of individual stocks presented above are obtained under the alternative hypothesis of random walk behavior of β_t. Estimates of this variance under the autoregressive hypothesis can be obtained for specified values of ρ with the help of equation (14).

Sample Selection Bias

Only those firms that satisfied a 100 percent data availability requirement over the respective intervals have been included in the results presented in Table I. The minimum data requirement induces a survival bias to the sample of firms. If the longevity of a firm were systematically related to the degree of nonstationarity in its market risk, the use of this sampling procedure would have induced a bias in the estimates presented in Table I.

We conducted the following tests to determine if the sample selection procedure has systematically affected the results presented in Table I. We computed the summary statistics for $\hat{\sigma}_v^2$ and t-values over each 75-month test interval for all firms which had 100 percent data available over the 150, 300, and 600-month intervals, respectively, in which the 75-month interval is contained. For example, out of 356 firms for which data are available over the 75-month interval from January 1926 to March 1932, 308 firms have data available for January 1926 to June 1938, 262 firms have data available for January 1926 to December 1950, and 127 have data available for the entire fifty-year period. Summary statistics for tests conducted over the 75-month period from January 1926 to March 1932 for the 356 firms were compared with the summary statistics for the 308, 262, and 127 firm subsets. Tabulation of results for these and other

[13] Since the power of the test declines (standard error of estimation of σ_v^2 increases) with the decrease in the number of observations used, the most powerful test (and most accurate estimate of σ_v^2) for any interval of time series is obtained by using all the data for the time interval. For example, if we were interested in estimating σ_v^2 over the entire 50-year period, the use of 50-year data provides a more accurate estimate of σ_v^2 than the two 25-year subseries can provide. Further, when the true value of σ_v^2 is greater than zero, the probability that the estimated value of σ_v^2 (and its "t-ratio") will be positive declines with the reduction in the number of observations used. This can be seen in the column headed "Percent > 0" in Table I.

Table II

Tests of Stability of Relative Risk of Portfolios

Data and Test Interval (75 mos. each)		Cross-Sectional Summary Statistics for Portfolio Size														
		1					10					25				
		No. of Portfolios	Mean	Median	Std. Dev.	Percent > 0	No. of Portfolios	Mean	Median	Std. Dev.	Percent > 0	No. of Portfolios	Mean	Median	Std. Dev.	Percent > 0
Jan. 1926–Mar. 1932		356				67	35				60	14				50
	$\hat{e}_i^2$		0.0186	.0072	.0495			.0010	.0008	.0033			.0003	−.0000	.0012	
	t-value		0.78	0.43	1.43			.43	.26	1.28			.27	−.02	.59	
Apr. 1932–June 1938		596				67	59				68	23				66
	$\hat{e}_i^2$		0.0048	.0019	.0202			.0006	.0003	.0017			.0003	.0002	.0004	
	t-value		0.66	0.54	1.18			.64	.46	1.27			.76	.53	1.04	
July 1938–Sept. 1944		711				84	71				86	28				96
	$\hat{e}_i^2$		0.0264	.0135	0.0535			.0061	.0022	.0091			.0021	.0013	.0019	
	t-value		1.81	1.49	1.80			2.23	2.00	2.06			2.93	3.16	1.96	
Oct. 1944–Dec. 1950		786				52	78				65	31				74
	$\hat{e}_i^2$		0.0047	.0008	.0374			.0003	−.0004	.0027			.0003	.0006	.0008	
	t-value		0.17	.04	1.06			.12	−.10	1.21			.45	.69	.85	
Jan. 1951–Mar. 1957		928				52	92				45	37				54
	$\hat{e}_i^2$		.0102	.0013	.1149			.0006	−.0006	.0044			.0002	.0001	.0014	
	t-value		0.26	.07	1.23			.13	−.18	1.24			.21	.10	1.16	
Apr. 1957–June 1963		898				48	89				34	35				37
	$\hat{e}_i^2$		−.0020	−.0036	.0334			−.0008	−.0008	.0018			−.0002	−.0002	.0011	
	t-value		−.08	−.28	1.10			−.34	−.54	.82			−.23	−.33	1.41	
July 1963–Sept. 1968		875				32	87				28	35				23
	$\hat{e}_i^2$		−.0114	−.0097	.0441			−.0014	−.0017	.0029			−.006	−.0010	.0008	
	t-value		−.30	−.44	.98			−.48	−.64	.85			−.69	−.81	.71	
Oct. 1969–Dec. 1975		1,105				54	110				59	44				61
	$\hat{e}_i^2$		.0064	.0011	.0251			.0009	.0002	.0021			.0004	.0002	.0009	
	t-value		0.51	.12	1.50			.58	.21	1.35			.66	.45	1.57	

		50					100				
Jan. 1926–Mar. 1932		7				71	3				67
	$\hat{a}_i^2$		.0002	.0003	.0003			.0002	.0002	.0002	
	t-value		.28	.08	.80			.85	.61	1.11	
Apr. 1932–June 1938		11				91	5				80
	$\hat{a}_i^2$		.0002	.0002	.0002			.0001	.0001	.0001	
	t-value		1.18	1.18	.94			1.54	1.51	1.33	
July 1938–Sept. 1944		14				93	7				100
	$\hat{a}_i^2$		.0013	.0010	.0013			.0006	.0006	.0003	
	t-value		3.55	3.27	2.40			3.36	3.54	1.21	
Oct. 1944–Dec. 1950		15				80	7				57
	$\hat{a}_i^2$		.0003	.0004	.0006			.0001	.0001	.0002	
	t-value		.77	.91	1.23			.58	.46	1.36	
Jan. 1951–Mar. 1957		18				61	9				56
	$\hat{a}_i^2$		.0003	.0003	.0007			.0000	.0001	.0004	
	t-value		.57	.39	1.38			.36	.28	1.44	
Apr. 1957–June 1963		17				35	8				50
	$\hat{a}_i^2$		−.0001	−.0000	.0004			−.0001	−.0001	.0002	
	t-value		−.13	−.08	1.38			−.15	−.51	1.01	
July 1963–Sept. 1969		17				18	6				25
	$\hat{a}_i^2$		−.0003	−.0002	.0006			−.0001	−.0003	.0003	
	t-value		−.48	−.42	.73			−.53	−.87	1.20	
Oct. 1969–Dec. 1975		22				16	11				55
	$\hat{a}_i^2$		.0004	.0002	.0006			.0002	.0001	.0003	
	t-value		1.26	.66	2.06			1.18	.62	2.07	

subintervals indicated that the sample selection procedure used does not seem to have introduced a bias into the estimates of nonstationarity.[14]

Tests on Portfolios

Several existing studies[15] have presented evidence on nonstationarity of the market risk of portfolios of common stocks and have concluded that the importance of nonstationarity declines with the increase in portfolio size. As with the existing studies of the behavior of the risk of individual common stocks, analysis of portfolios usually involves analysis of OLS estimates of risk from two or more adjacent segments of time series. Table II contains estimates of the variance of changes in the market of risk of portfolios of various sizes over the eight 75-month intervals. Equally weighted portfolios are formed by random sampling of stocks from the CRSP file. Individual stocks included in the portfolios are those for which all the 75 months of data are available.

The variance of changes in the market risk of portfolios decreases as the portfolio size increases. This is true of every one of the eight 75-month intervals. For example, during July 1938–September 1944, the average variance declined from 0.0264 for individual stocks to 0.0051, 0.0021, 0.0013 and 0.0005 for 10, 25, 50 and 100 stock portfolios respectively. The median values are slightly lower than the mean values.

If changes in risk of stocks are cross-sectionally independent, we would expect the variance of the changes in the market risk of portfolios to decline in inverse proportion to the number of stocks in the portfolio. A graph of estimated variance on size of portfolios indicated that in every one of the eight periods examined, the observed decline is approximately proportional to the inverse of the portfolio size.[16] It may be inferred from this evidence that the changes in market risk of individual stocks are, for the most part, cross-sectionally independent. They cannot be completely independent since the risk of the market portfolio is fixed at 1.

The period from July 1938 to September 1944 exhibits the greatest degree of nonstationarity in the market risk for individual stocks; during the four 75-month intervals beginning October, 1944, the risk of individual stocks shows little nonstationarity on average. The same is also true of the portfolio risk; the periods of high nonstationarity in the risk for individual stocks are also the periods of high risk uncertainty for the portfolios.

[14] This table is available from the author on request.

It may appear to be surprising that the stocks with high nonstationarity in their risk are no more likely to be delisted than those having relatively stable risk. However, a high degree of nonstationarity only implies a higher chance of a change in the market risk, which can be either positive or negative. If we assume that the high-risk stocks are more likely to be delisted than the low-risk stocks, the higher probability of delisting of stocks which undergo a positive change in their market risk is balanced by the lower probability of delisting of those which undergo a negative change. On the whole, therefore, higher nonstationarity need not imply higher probability of delisting.

[15] See Blume [2]; Jensen [11]; Beaver, Kettler and Scholes [1].

[16] The graphs are available from the author on request.

Concluding Remarks

The purpose of this paper has been to present unbiased and consistent estimates of variance of changes in the market risk individual common stocks and their portfolios. The estimates of variance depend on the hypothesis about the behavior of risk over time. Two specific alternatives to the null hypothesis of stationarity are considered: First, the market risk follows a random walk over time, and second, it follows an autoregressive process. The estimates of variance are presented for the random walk hypothesis. The procedure for obtaining estimates for the more general autoregressive hypothesis under a prespecified value of serial correlation coefficient, ρ, is also presented.

The null hypothesis of stationary risk, $\sigma_v^2 = 0$, is soundly rejected for the population of stocks during the fifty-year period 1926-75 and both the 25-year subperiods. There are strong signs of nonstationarity in three of four 150-month subperiods and in three of eight 75-month subperiods. The average level of nonstationarity varies from one subperiod to another, being high during 1926-50 and low during 1963-75.

Efficient estimation of market risk requires estimates of step variance σ_v^2. Unbiased and consistent estimates of σ_v^2 for individual stocks and portfolios over any given time series given in this paper can be combined with efficient estimation techniques already available in the literature (see Footnote 1) to obtain better estimates of market risk.

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Co-Skewness and Capital Asset Pricing

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I. Introduction

VIRTUALLY ALL OF THE early studies of the Sharpe-Lintner capital asset pricing model (CAPM) found the predicted linear relationship between return and the non-diversifiable risk of risky assets, generally represented by common stocks listed on the New York Stock Exchange (NYSE). However, they also found that this return-risk relationship seemed to imply for most periods a riskless market rate of return substantially above any reasonable measure of the actual risk-free rates of return. Recent papers point to a similar result if the market portfolio of risky assets is represented by an appropriately weighted portfolio of common stocks and bonds instead of common stocks alone.[1]

Thus, it is noteworthy that a study by Kraus and Litzenberger finds that a measure of co-skewness can be used as a supplement to the co-variance measure of risk to explain the returns on individual NYSE stocks, and in the process to explain the otherwise observed discrepancies between these returns and the returns on NYSE stocks as a whole.[2] In other words, they extend capital asset pricing theory to incorporate the effect of skewness in return distributions, making the assumption that investors have a preference for positive return skewness in their portfolios (and therefore positive or negative co-skewness in individual assets depending on the skewness in the market portfolio).[3] As a consequence Kraus and Litzenberger are apparently able to explain observed returns in the stock market without the substitution of a non-observable zero-beta construct for the risk-free rate. Kraus and Litzenberger assume that just as investors are averse to variance in their portfolios, and therefore beta in individual assets, they prefer positive skewness in their portfolios. Hence, since they also assume that all investors hold the market portfolio, investors would be willing to pay a premium for assets which possess positive co-skewness with the market if

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[1] See Friend, Westerfield and Granito [4] and Friend and Westerfield [5]. These papers also raise serious questions about exclusive reliance on nondiversifiable risk in measuring the overall riskiness of assets.

[2] See Kraus and Litzenberger [6].

[3] Their basic approach is to expand a utility function beyond the second moment in a Tayler Series and to examine skewness effects. They do not consider higher order effects. Assuming separation all investors choose the market portfolio (in equilibrium). The market portfolio is not mean-variance efficient but is efficient with respect to the utility functions that lead to separation. Thus the Kraus-Litzenberger model is different from but can be regarded as an extension of the earlier forms of the CAPM.