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Accuracy of Exchange Valuation Rules

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Various approaches to valuation have often been regarded as empirical proxies for a common, unobservable, theoretical construct of value.[1] From this viewpoint, the main objective of research in valuation is to determine how well each method approximates the underlying value. There is no unique measure of how "good" a proxy is, and several attributes such as relevance to decisions, objectivity, freedom from bias, and cost have been examined in the accounting literature. Most analyses have been centered on historical cost, general price level, current valuation, and their variations. Since little attention has been paid to the underlying analytical structure of the above-mentioned valuation rules, a rigorous comparative study of their attributes has not been possible. The purpose of the present study is to introduce a new approach to the comparative analysis of valuation systems. This is accomplished in three steps.

In Section 1, a general scheme for algebraic representation of a family of valuation rules, called the "exchange valuation set" is presented. Historical cost, general purchasing power, and current valuation rules are seen to be three of the large number of members that belong to this family. Study of valuation rules by means of this structure offers two distinct advantages. First, it facilitates an analysis of the differences between valuation rules in quantitative rather than merely qualitative terms. The choice of a valuation rule is not so

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[1] See Nelson [1973].

much a matter of selecting from radically different accounting principles as selecting a point on a continuum. For example, the distinction between general price level and current value accounting is seen to be the difference between the number of price indexes used to adjust the historical price of assets; the number is one for the former and many for the latter.[2]

Second, this algebraic structure of valuation rules suggests ways of quantifying certain attributes of valuation rules. In Section 2, I suggest that exchange valuation rules be compared along three dimensions – accuracy or distance from the underlying concept of value, cost of the accounting system, and objectivity or hardness. In other words, each valuation rule can be represented by a point in a three-dimensional space, so its relative accuracy, cost, and objectivity can be compared and trade-offs among the attributes can be made to facilitate choice. This representation of valuation rules will help us move toward application of the principles of cost-benefit[3] analysis to the choice of valuation systems, a goal long cherished but rarely realized.

Sections 2, 3, and 4 of the study consist of a detailed scrutiny of the accuracy of the valuation rules belonging to the exchange valuation set. Two measures of accuracy, mean squared error and bias, are used. Six propositions state the major results on accuracy of exchange valuation rules. Other attributes of exchange valuation rules and the problem of choice of valuation rules are briefly discussed in Section 5. Concluding remarks are in Section 6.

1. Family of Exchange Valuation Rules

A valuation rule is a procedure for assigning numbers (of monetary units) to economic resources. Exchange valuation rules are those rules in which the number assigned to each resource is the number of monetary units for which it has been, or can be, exchanged. Exchanges can take place under a variety of conditions and, in each case, the number of monetary units exchanged for the resource can be estimated in several ways. For example, a machine could be bought, sold, leased in, or leased out in exchange for money or other goods. In each type of transaction, the actual number of dollars that change hands is only one of several possible appraisals. The reason is that the time of valuation does not necessarily coincide with the time of exchange. When exchange and valuation occur at different times, several esti-

[2] A useful distinction is drawn between the factors that bring about changes in general price level and in relative price of individual assets. However, changes in general price level are not observationally independent of changes in relative prices. It is this observational dependence of general price level on relative prices that substantially eliminates the need to maintain a sharp distinction between the two in discussion of valuation procedures.

[3] See Horngren [1975] for an argument for application of cost-benefit analysis to the problem of valuation.

mators of the number of dollars become candidates for consideration. Combinations of the different types of exchange transactions with the different estimators of the number of dollars exchanged for each type of transaction constitute a family of valuation rules. I will call this family the "exchange valuation set" and analyze the interrelationships among the members of this family.

The exchange valuation set includes most systems of valuation encountered in accounting literature and in current practice. Historical cost, replacement cost, realizable value, and general purchasing power rules are all members of this set.

The set can be defined to include the hybrid systems such as "lower of cost or market." However, hybrid systems do not easily yield to the formal analytical approach of this study and are therefore excluded. Discounted cash flow (*DCF*) valuation can also be included here by viewing the *DCF* valuation as an estimate of the number of monetary units for which an asset is exchanged, adjusted for the time value of money. Since the line between *DCF* and realizable value rules is not well defined,[4] and replacement or realizable values are sometimes used as surrogates for *DCF*, separate consideration of *DCF* does not seem necessary.[5]

Members of the exchange valuation family can be cross-classified by two criteria, the direction of exchange and the level of aggregation of price indexes used for estimating the number of dollars for which a good is exchanged.

Direction of Exchange. Since the exchange price of each good can be taken from either entry or exit transactions, the direction of exchange criterion bisects the exchange valuation family into two subsets. These subsets are equal, disjoint, collectively exhaustive, and isomorphic. If a relationship holds between any two elements of one subset, it also holds for the corresponding elements of the other subset. In the present study, we restrict ourselves to the examination of relationships among the valuation rules included within each subset. Since

[4] Canning [1929, pp. 206–47] proposed the use of direct valuation (discounted exit price) for current assets and indirect valuation (entry price of embodied services) for the long-lived tangible assets. Staubus (in Sterling [1971, p. 65]) places even greater emphasis on similarity between *DCF* and net realizable value: "It [*DCF*] relies upon the same type of evidence as does net realizable value, viz., a past transaction price, so surely it cannot be said to be any less objective than net realizable value." At the very least, the future in *DCF* extends beyond the future in net realizable value and therefore the former may be said to encompass the latter.

[5] Another related valuation rule, "value to the firm" (see Solomon [1971, p. 111]), need not be considered as a distinct member of the exchange valuation set. There are two interpretations of the term "value to the firm." One is the *DCF* concept representing the present value of associated cash flows in its present use. The second is the idea that replacement and net realizable values are the bounds for the "value to the firm," and within these bounds, this value is indeterminate. The determinate bounds of the range are elements of the exchange valuation set. Within the range, "value to the firm" is identical with the *DCF* valuation.

the differences between exit and entry prices arise from market imperfections such as the cost of transactions, information, and transportation, and from the problems of the indivisibility and composition of assets into an integrated whole, an analytical examination of the relationship between the entry and exit subsets of the exchange valuation family demands detailed specification of such imperfections, a task too extensive for the present study.

Level of Aggregation. The second criterion used for classifying the exchange valuation set is the level of aggregation of price indexes used for valuation. Starting with the historical estimator through single-index to multi-index price valuation rules, a large number of alternative methods are available which differ with respect to the configuration and the level of aggregation of price indexes applied to historical prices in order to obtain the estimate of current value. The general price level valuation involves the use of a single-index estimator while various proposals for current value systems involve the use of multi-index estimators. Since the historical valuation uses historical estimates without any adjustment for price changes, it could be viewed as the "zero-index" estimator. We shall refer to this criterion of distinguishing among the systems of exchange valuation as the level of aggregation criterion.

The direction of exchange and the level of aggregation criteria are used to cross-classify the exchange valuation set. Note that for each level of aggregation, there is one element each in entry and exit valuation subsets. The corresponding elements of the two subsets differ in only two respects. Entry values are obtained by applying the appropriate number of indexes of *entry prices* to the historical *entry* price; the *exit values* are obtained by applying the indexes of *exit prices* to the historical *exit* price.[6] Since the development of entry and exit subsets is parallel and we are concerned with comparisons within each subset and not between them, we shall frequently find it convenient to drop the qualifiers entry and exit without causing any misunderstanding.

ALGEBRAIC STRUCTURE OF THE EXCHANGE VALUATION SET

Exchange valuation rules can be clearly and unambiguously represented by simple algebraic notation. Suppose under a given rule, the valuation of a given basket of n distinct goods is P^0 at time 0 and P^1 at time 1. The valuation at time 1 can be given by relative change $R = (P^1 - P^0)/P^0$. Given P^0 and R, valuation at time 1, P^1 can easily be obtained, $P^1 = P^0 (1 + R)$. Characterization of valuation rules by the relative change in valuation, R, is convenient for the subsequent theory which is developed in terms of price indexes, also expressed in terms of relative changes.

[6] In a system of exit price valuation, historical exit prices will replace the historical entry prices.

R is the generic symbol for valuations and two modifiers are added to identify a specific valuation rule. R_{ki} represents a valuation rule which uses k different price indexes to adjust the beginning period valuation of all n goods in the basket. Since there can be more than one, say L_k, ways of forming k price indexes for n goods, R_{ki} represents the valuation obtained by using the ith of these L_k index configurations. We shall now show how this notation can be used to represent each element of the exchange valuation set.

Historical Valuation. It is easily seen that for the historical valuation rule, P^1 is equal to P^0, and therefore $R = (P^1 - P^0)/P^0$ is zero. The number of price indexes necessary for making this adjustment is zero, and there is only one way of doing so. We shall represent historical valuation by $R_{0,1}$ and set it equal to zero:

$$R_H \equiv R_{0,1} = 0. \tag{1}$$

Note that P^0 and P^1 are valuations of the same basket of goods at two different instants under the same valuation rule. These assumptions are maintained throughout the paper.

General Price Level Valuation. The general price level valuation requires the use of a single price index to adjust the beginning period valuation P^0. There is only one way of partitioning a set of goods to construct one price index. Therefore, the general price level estimator is denoted by $R_{1,1}$. $R_{1,1}$ is a weighted average of fractional price changes for goods included in a standard basket used for defining the general price level. Let G, an ordered set of n distinct goods, denote the basket. Also, let $\mathbf{q} = (q_1, q_2, \ldots, q_n)'$ be the quantity of each good in appropriate units and $\mathbf{p}^t = (p_1^t, p_2^t, \ldots, p_n^t)'$ be the unit exchange price of each good at time t $(t = 0, 1)$. If $\mathbf{r} = (r_1, r_2, \ldots, r_n)'$, (prime denotes transposition of vectors) is the relative price change from $t =$ 0 to 1 for each good, then:

$$r_i = (p_i^1 - p_i^0)/p_i^0 \text{ for } i = 1, 2, \ldots, n. \tag{2}$$

The average relative price change for basket G from time 0 to 1 is:

$$\begin{aligned} R_{GPL} \equiv R_{1,1} &= \frac{\sum_{i=1}^{n} p_i^1 q_i - \sum_{i=1}^{n} p_i^0 q_i}{\sum_{i=1}^{n} p_i^0 q_i} \\ &= \sum_{i=1}^{n} \left(\frac{p_i^0 q_i}{\sum_{i=1}^{n} p_i^0 q_i} \right) \left(\frac{p_i^1 - p_i^0}{p_i^0} \right) \qquad (3) \\ &= \sum_{i=1}^{n} \omega_i r_i = \omega' \mathbf{r} \end{aligned}$$

where ω is the $(n \times 1)$ vector of relative weights of each good based on prices at time 0, used for the construction of the price index. The general price level valuation, $R_{1,1}$, is the weighted average of the fractional change in the price of individual goods.

Note that the quantity of goods q remains unchanged from time 0 to 1. This assumption is maintained throughout the study in order to focus attention on valuation and to exclude the consideration of the flow of goods. Also note that the vector q of quantities and the resulting vector ω of relative weights are not specific to a firm. They are determined by whatever mechanism is used to construct the general price index. For the three most commonly mentioned general price indexes, that is, Consumer Price Index, Wholesale Price Index, and GNP Deflator, the task is performed by the agencies of the federal government.

Specific Price or Current Valuation. There are two equivalent ways of calculating the current value of a good. One is to multiply the current price of each good by its quantity; another is to multiply the historical valuation by the relative change in price "index" for that good. If the price index is constructed for a single good, the relative change in the index is identical to the relative price change for that good. In the n-goods economy, current valuation is equivalent to using n price indexes to adjust the beginning period valuation of each good. Unlike the general price-level valuation, where a single index is used, current valuation needs one index for each good. Since there is only one way of partitioning n goods to construct n price indexes, this valuation rule is denoted by $R_{n,1}$. If w is the $(n \times 1)$ vector of relative weights in the basket of goods being valued, it is easily shown that:

$$R_{CV} \equiv R_{n,1} = \mathbf{w}'\mathbf{r} \tag{4}$$

where

$$w_i = \frac{p_i^0 q_i^*}{\sum_{i=1}^{n} p_i^0 q_i^*} \quad \text{for} \quad i = 1, 2, \cdots, n.$$

Note that q* in this case is the vector of quantities in the basket being evaluated and not the basket used for construction of the price index. Accordingly, the weights w are, in general, different from ω, which are based on q.

Multi-Index Valuation. Valuation rules which use between one and n (which is the number of distinct goods) price indexes are termed multi-index valuation. Sandilands' Report [1975], for example, suggests the use of nineteen specific price indexes for valuation of machinery and plant assets. Each price index is a weighted average of price changes for a subset of goods. Use of k price indexes, for example, requires that the set of n goods be partitioned into k subsets, a weighted average price return be calculated for each subset, and the product of total historical price in a subset and its price index be added for all subsets.[7] This sum is the multi-index valuation.

[7] See Appendix A for the definition of partition and fineness of partitions.

Multi-index valuation is different from historical ($R_{0,1}$), general price level ($R_{1,1}$), and specific price valuation ($R_{n,1}$) rules in the sense that it is not a single element, but a subset of exchange valuation rules. In fact, all except three elements of the exchange valuation set are specific price index valuation rules. The number of price indexes used, k, can take any value from 2 to $n - 1$. For each value of k, there are:

$$L_k = \sum_{j=0}^{k-1} \frac{(k-j)^n}{j!(k-j)!}(-1)^j \tag{5}$$

distinct ways of forming k price indexes for n goods by using different combinations. The general price level ($k = 1$) and specific price ($k = n$) valuation rules discussed earlier are single- and n-index special cases of specific price index valuation. $L_1 = L_n = 1$ because there is only one way each of constructing one and n price indexes for n goods. If these two are included in the class of multi-index valuation, the total number of elements in this class, $M(n)$, is given by:

$$M(n) = \sum_{k=1}^{n} L_k. \tag{6}$$

Even for moderate values of n, $M(n)$ is a large number.

Multi-index valuation using k indexes is not unique. It depends not only on the number of price indexes, but also on which goods are combined into each price index. We shall assume that the L_k partitions of set G have been arbitrarily indexed by $i = 1, 2, \cdots, L_k$. Then R_{ki} can be used to denote a k-index estimator which uses the ith of the L_k-partitions of G. This partition is denoted by π_{ki}.

Without loss of generality, let $(\mathbf{r}_1' \mid \mathbf{r}_2' \mid \cdots \mid \mathbf{r}_k')'$, $(\omega_1' \mid \omega_2' \mid \cdots \mid \omega_k')'$, and $(\mathbf{w}_1' \mid \mathbf{w}_2' \mid \cdots \mid \mathbf{w}_k')'$ be the partitions of vectors $\mathbf{r}$, ω, and $\mathbf{w}$, respectively, which correspond to π_{ki}. From the definition of specific price index valuation given above, it is easily shown that:

$$R_{ki} = \sum_{i=1}^{k} \left(\frac{\mathbf{w}'_u \mathbf{e}}{\omega'_u \mathbf{e}} \omega'_u \mathbf{r}_u \right) \tag{7}$$

$u = 1$

where $\mathbf{e}$ is a vector of unit elements and appropriate length. This expression is a generalization of equations (4) and (5) and can be reduced to the latter by setting $k = 1$ and n, respectively, and using the corresponding partitions of w, ω, and r. Equation (7) is therefore the general algebraic representation of exchange valuation rules.

A traditional approach to the analysis of valuation rules employs a nominal classification of valuation rules which emphasizes their qualitative differences.[6] The approach used here, on the other hand, empha-

[6] See, for example: "I agree with Professor Bell that price-level accounting based on a general purchasing-power index does not have any relationship to current costs, or with current values. Of course, price-level accounting does not purport to have any such relationship" (Catlett in Sterling [1971, p. 37]). Mautz (in Davidson and Weil [1977, pp.

sizes the qualitative similarities and quantifies their differences so each valuation system can be represented in a multidimensional space by a point. The framework presented here facilitates a complete analysis of all valuation rules in the set. I hope that this view of exchange valuation rules as distinct but related members of a family will at least prove as useful in itself as the analysis of properties of the set which follows.

2. *Accuracy of Exchange Valuation Systems*

If each valuation rule is viewed as a method to approximate an underlying theoretical construct of value, the ability of the valuation rule to approximate or surrogate the principal quantity is obviously an important attribute of the rule.[9] This attribute is accuracy. If we can define a measure of accuracy and obtain data on the principal and the alternative proxies, it would be an easy task to evaluate how well each proxy is able to approximate the principal. Several measures of accuracy can be devised, and data on proxies can be collected. However, the principal itself is rarely observable. Empirical probes into the comparative accuracy of valuation methods have been blocked by the unavailability of data on the principal quantity.

But the obstacle is not insurmountable. We can hypothesize a structure for the behavior of values of assets and analytically evaluate the goodness of various valuation rules relative to this structure. The structure can then be changed and the proxies reevaluated. This procedure will yield an understanding of the relationships that exist among the alternative proxies (valuation methods) under various structures, so that we can identify those properties and relationships of alternative valuation rules which are not dependent on the structure of values. The next two sections of this paper are devoted to the investigation of the accuracy of the exchange valuation system relative to a specified structure of values.

In future studies, the descriptive validity of alternative value structures can be assessed against the macroeconomic price data which are already available. There have been few attempts in the past to use the macroeconomic price series for this purpose, and accounting empiricists have tended to concentrate on the output of the accounting system alone. Obviously, this output can yield little information about the ability of alternative proxies to approximate the underlying construct of value.

1-7]) takes a similar view. Ijiri [1975, chap. 7], who refers to them as *general and specific price adjustments*, clearly recognized the family relationship and introduced the analytical approach used in this paper.

[9] See Ijiri [1967, Chap. 1] on principal-surrogate relationships and their importance in accounting: " . . . the fact that a representation is imperfect does not necessarily mean that it is totally useless . . . . This is fortunate for those who design accounting information systems since if an imperfect representation were totally useless, it would be virtually impossible to develop any workable accounting information system."

The present study may be viewed as a preliminary step toward building a theory of comparative valuation. I derive the relationship among a broad class of proxies or valuation methods, called exchange valuation methods, by hypothesizing an underlying structure of values. Some interesting relationships are observed. For example, contrary to popular belief, the use of a larger number of price indexes does not necessarily provide a "better" proxy of value.

3. Accuracy of Exchange Valuation Rules—Individual Firms

In Section 1, I developed a scheme of representing the exchange valuation rules as linear aggregates of quantities of goods weighted by prices. They differ from one another with respect to the weights used for aggregation. Any exchange valuation rule can be represented by R_{ki}, where the ith of the L_k possible configurations of k price indexes is used. I have already shown (Section 4) that the underlying value, the 1
principal aggregation R^*, is also the n-index valuation $R_{n,1}$. In this section, the ability of various elements of the exchange valuation set to surrogate the principal aggregation is examined.

The accuracy of R_{ki} as an estimator of R^* can be defined in several ways. I have used the expected value of the difference between R_{ki} and R^* (bias) and the expected value of the squared difference (mean squared error or *MSE*) as two measures of accuracy in this study. Bias and *MSE* can be shown to be appropriate for linear and quadratic loss functions, respectively.[10]

Expressions for bias and mean squared error association with each element of the exchange valuation set have been derived in Appendix A and are summarized in columns 3 and 4 of table 1. In order to define these properties, we need to specify the first and second moments of the probability distribution of relative price changes for each of the n goods. Let vector μ and nonnegative definite symmetric matrix Σ denote the mean and covariance of n relative price changes. All other symbols used in the expressions have already been defined in Section 1.[11]

Bias. As an estimator of the current value, the assumed principal in this study, the historical cost valution is biased by an amount $-\mathbf{w}'\boldsymbol{\mu}$, the weighted average of mean relative price changes for the firm. Whether the bias is negative or positive depends on the relative weights

[10] Ijiri [1967; 1968] suggested that the linear aggregation coefficient be used as a measure of how well one aggregate approximates another. It is easily shown that *MSE* is a linear monotonic function of the linear aggregation coefficient, and therefore it provides all information that the latter measure does. MSE is also dependent on the variances of the principal and surrogate aggregates and their mean difference. All results of the following analysis can be restated in terms of the linear aggregation coefficient.

[11] Note that we have hypothesized a structure of underlying values by specifying the mean (μ) and covariance ($\sum$) of relative price changes and vector w of relative weights of assets in the portfolio of the firm.

TABLE 1
Accuracy of Exchange Valuation Rules

Valuation System		For a Given Firm w: Bias $E_{\tilde{r}}(R - R^*)$	For a Given Firm w: Mean Squared Error $E_{\tilde{r}}(R - R^*)^2$	Economy-wide Averages: Bias $E_{\tilde{w}}E_{\tilde{r}}(R - R^*)$	Economy-wide Averages: Mean Squared Error $E_{\tilde{w}}(E_{\tilde{r}}(R - R^*)^2)$
Historical	$R_{0,t} = 0$	$-\mathbf{w}'\mu$	$\mathbf{w}'\Sigma\mathbf{w} + \mathbf{w}'\mu\mu'\mathbf{w}$	$-\omega'\mu$	$\frac{\rho - 1}{\rho}\,\omega'(\Sigma + \mu\mu')\omega + \frac{1}{\rho}\,\omega'(\sigma + \dot{\mu})$
Current Price-Level or Single-Index	$R_{1,t} = \omega'\mathbf{r}$	$(\omega - \mathbf{w})'\mu$	$(\omega - \mathbf{w})'\Sigma(\omega - \mathbf{w}) + (\omega - \mathbf{w})'\mu\mu'(\omega - \mathbf{w})$	0	$\frac{1}{\rho}\,(\omega'(\sigma + \dot{\mu}) - \omega'(\Sigma + \mu\mu')\omega)$
k-Index	$R_{ki} = \sum_{\alpha=1}^{k} \frac{\omega_\alpha'\mathbf{r}_\alpha}{\omega_\alpha'\mathbf{e}}\,\mathbf{w}_\alpha'\mathbf{e} = (\omega^{ki})'\mathbf{r}$	$\left[\begin{pmatrix} \frac{\mathbf{w}_1'\mathbf{e}}{\omega_1'\mathbf{e}}\cdot\omega_1 \\ \vdots \\ \frac{\mathbf{w}_k'\mathbf{e}}{\omega_k'\mathbf{e}}\cdot\omega_k \end{pmatrix} - \mathbf{w}\right]'\mu = (\omega^{ki} - \mathbf{w})'\mu$	$(\omega^{ki} - \mathbf{w})'\Sigma(\omega^{ki} - \mathbf{w}) + (\omega^{ki} - \mathbf{w})\mu\mu'(\omega^{ki} - \mathbf{w})$	0	$\frac{1}{\rho}\left\{\omega'(\sigma + \dot{\mu}) - \sum_{\alpha=1}^{k} \frac{\omega_\alpha'}{\omega_\alpha'\mathbf{e}}\,(\Sigma_{\alpha\alpha} + \mu_\alpha\mu_\alpha')\omega_\alpha\right\}$
Current-Value or n-Index	$R_{n,t} = \mathbf{w}'\mathbf{r}$	0	0	0	0

$$\omega^{ki} = \begin{bmatrix} \frac{\mathbf{w}_1'\mathbf{e}}{\omega_1'\mathbf{e}}\omega_1 \\ \vdots \\ \frac{\mathbf{w}_k'\mathbf{e}}{\omega_k'\mathbf{e}}\omega_k \end{bmatrix}$$ where k is the number of subsets in the partition of G and ith of the L_k k-partitions is used for valuation.

σ = vector of diagonal elements of Σ.
$\dot{\mu}$ = vector of squared elements of μ.

and mean relative price changes for individual assets in the firm's portfolio. When all relative weights and mean price changes are positive, the bias is obviously negative—a familiar complaint about historical cost valuation under inflation.

For general price level or single-index valuation, the bias is given by $(\omega - \mathrm{w})'\mu$, the product of mean price changes and the difference between weights of individual assets in the portfolios held by the firm and used for construction of the index. This bias is not necessarily smaller than the bias for historical valuation, unless we limit all relative weights and price changes to positive quantities. The same is true of the general multi-index estimator R_{kl}, whose bias is given by $(\omega^{kl} - \mathrm{w})'\mu$.[12] Since the n-index estimator $R_{n,1}$ is identical to the principal R^*, its bias is zero by definition.

A counterexample is sufficient to prove the validity of the above assertions about the bias of exchange valuation rules. Consider an economy with four goods a, b, c, and d with the expected relative price change μ and covariance matrix Σ given by:

$$\mu' = (.05, .10, .08, .01)$$

and

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}.$$

Suppose that the relative weights used for index construction are given by $\omega' = (0.3, 0.05, 0.5, 0.15)$. The exchange valuation set for the four-good economy consists of sixteen elements with one element each for zero-, one-, and four-index valuation, seven for two-index valuation, and six for three-index valuation. These index configurations are listed in table 2. Bias of each method of valuation for a specific firm which has the four assets in relative proportions given by $\mathrm{w}' = (0.1, 0.3, 0.15, 0.45)$ is shown in table 2. A perusal of the numbers in column 4 shows that the bias associated with multi-index valuation is often greater than the bias of single-index valuation, both in absolute and algebraic terms.

If bias were to be used as a measure of accuracy, it would be desirable to rank the elements of the exchange valuation set by the amount of bias. Since the changes in bias as the number of indexes is increased from zero toward n depend on the relative weights of assets in the portfolio of the firm being analyzed, no ranking of valuation rules by their bias is possible. Thus the following proposition: *Proposition 1*. The bias of exchange valuation systems with respect to a specific firm does not necessarily decrease with an increase in either the number or the fineness[13] of price indexes used.

[12] See the note to table ~~2~~ 1 for the definition of ω^{kl}.

[13] See Appendix A for the definition of fineness.

TABLE 2

A Numerical Example of the Bias of Exchange Valuation Rules

No. of Indexes k	No. of Index Configurations L_k	Index Configurations	Bias for the Specific Firm w	Average Bias
0	1		−.0515	−.0615
1	1	1. $(abcd)$	0.001	0
2	7	1. $(a\ bcd)$	0.0133	0
		2. $(b\ acd)$	0.0201	0
		3. $(c\ abd)$	−0.0029	0
		4. $(d\ abc)$	−0.0082	0
		5. $(ab\ cd)$	0.0097	0
		6. $(ac\ bd)$	−0.0099	0
		7. $(ad\ bc)$	0.0066	0
3	6	1. $(ab\ c\ d)$	−0.0121	0
		2. $(ac\ b\ d)$	0.0002	0
		3. $(ad\ b\ c)$	0.0107	0
		4. $(a\ bc\ d)$	−0.0052	0
		5. $(a\ bd\ c)$	−0.0101	0
		6. $(a\ b\ cd)$	0.0218	0
4	1	1. $(a\ b\ c\ d)$	0	0

Set G = (a, b, c, d), $n = 4$.
Index weights ω' = (0.3, 0.05, 0.5, 0.15).
Relative weights for the firm $\mathbf{w}'$ = (0.1, 0.3, 0.15, 0.45).
Expected change in relative prices, μ' = (.05, .1, .08, .01).

Mean Squared Error (MSE). The second measure of accuracy of valuation rules we consider is the *MSE* with respect to the n-index (current value) estimator. Expressions for *MSE* of various exchange valuation systems are given in column 4 of table 1. The *MSE* of $R_{n,1}$ is zero by definition. For other valuation rules, it is the sum of squared bias and variance. I have already shown that the bias term does not necessarily get smaller with the number or fineness of price indexes used; variance does not either.

Proposition 2. The mean squared error of exchange valuation rules with respect to a specific firm does not necessarily decrease with an increase in either the number or fineness of the price indexes used. The proof of this proposition by a counterexample is given in Sunder [1976] and summarized in figure 1 and column 4 of table 3. The example used is the same as the one used earlier to illustrate the behavior of bias, except that μ has been set equal to 0. The figure displays the mean squared error associated with every possible exchange valuation rule (index configuration) for a specific firm. Each valuation rule is represented by a node and is connected by a line to a strictly coarser index configuration to the left, and to a strictly finer configuration to the right. There is only one method each of using zero, one, and four price indexes in a four-good economy. Mean squared error of historical valuation, 0.40, increases to 0.82 when general price level (simple-index) valuation is used. There are seven possible schemes for forming two indexes of four goods, and for one of these, the mean squared error is even higher at 1.36. Similarly, several three-index valuation rules have higher *MSE* than some two-index rules.

Several other interesting propositions about the properties of exchange valuation rules with respect to specific firms can also be derived from their algebraic definitions.

Proposition 3. A sufficient condition for the mean squared error of an exchange valuation rule when it is applied to a specific firm to be zero is that the vector $\tilde{w}$ of relative weights for the firm be equal to the vector ω of index weights. The proposition also applies to bias. Proof is direct

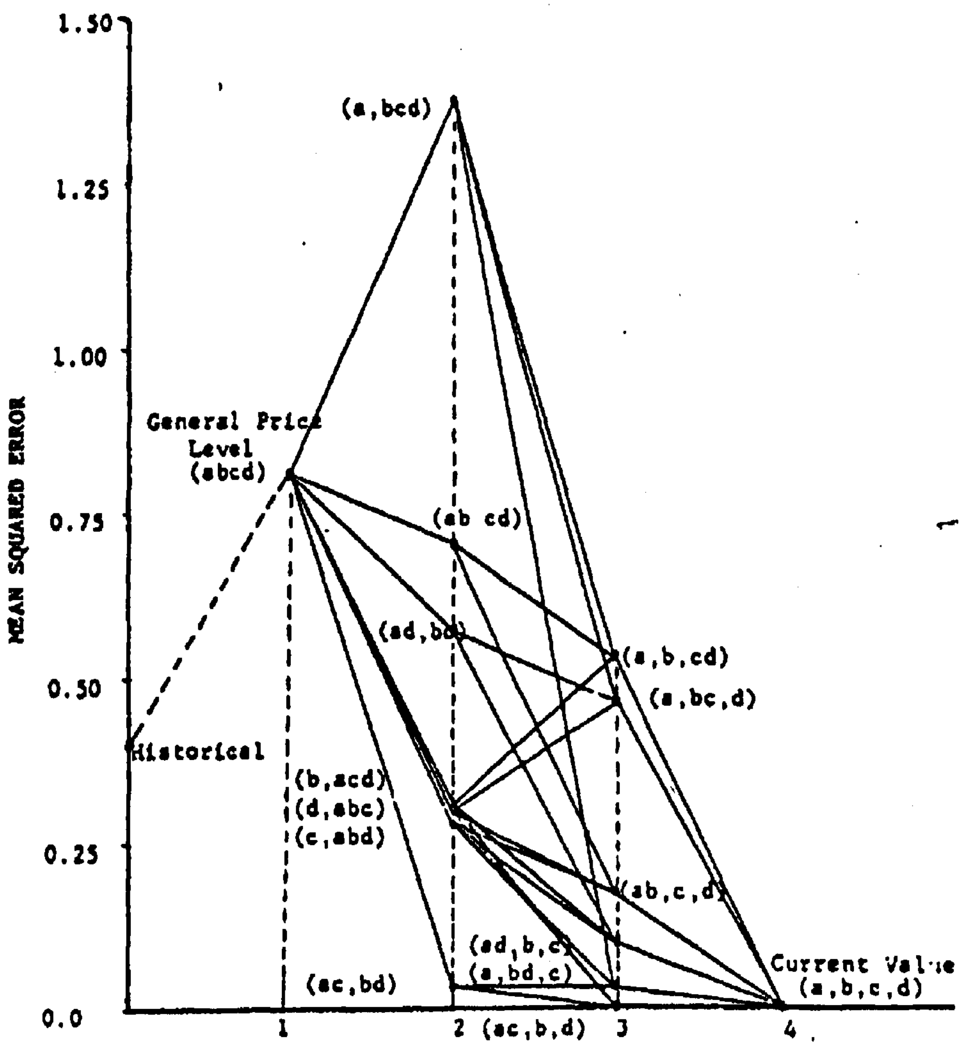

FIG. 1.—Mean squared error of exchange valuation rules for a specific firm X, w (0.1, 0.3, 0.15, 0.45). Letters in parentheses indicate the price index configurations for four goods, G (a, b, c, d). Valuation systems comparable with respect to fineness are joined by straight lines.

TABLE 3

A Numerical Example of the Accuracy of Exchange Valuation Rules

No. of Indexes k	No. of Index Config-urations L_k	Index Configurations	MSE for the Specific Firm w	AMSE $\times\, \rho$
0	1		0.40375	$1.35625\rho + 1.61875$
1	1	1. $(abcd)$	0.8225	1.61875
2	7	1. $(a\ bcd)$	1.3587	0.8661
		2. $(b\ acd)$	0.3108	1.4526
		3. $(c\ abd)$	0.2796	0.2625
		4. $(d\ abc)$	0.3027	1.3176
		5. $(ab\ cd)$	0.7107	0.7632
		6. $(ac\ bd$	0.0319	1.2187
		7. $(ad\ bc)$	0.5766	0.4682
3	6	1. $(ab\ c\ d)$	0.1769	0.1286
		2. $(ac\ b\ d)$	0.0002	1.125
		3. $(ad\ b\ c)$	0.1067	0.15
		4. $(a\ bc\ d)$	0.4699	0.3182
		5. $(a\ bd\ c)$	0.0316	0.0942
		6. $(a\ b\ cd)$	0.5338	0.6346
4	1	1. $(a\ b\ c\ d)$	0	0

Set $G = (a, b, c, d), n = 4$.
Index weights $\omega' = (0.3, 0.05, 0.5, 0.15)$.
Relative weights for a specific firm $w' = (0.1, 0.3, 0.15, 0.45)$.

by substituting $\omega = w$ in (A.7) and (A.11). This condition for perfect accuracy, $w = \omega$, is too severe and not necessary. Perfect accuracy can be obtained under much weaker conditions.

Proposition 3.1. A necessary and sufficient condition for the mean squared error of an exchange valuation rule R_{kl} for a specific firm to be zero is that for each of k price indexes, normalized subvector of relative weights for the firm, $w_u/w'_u e$ be equal to the corresponding normalized subvector of index weights, $\omega_u/\omega'_u e$ $(u = 1, 2, \cdots, k)$. The condition is sufficient but not necessary for bias to be zero. Sufficiency can be confirmed by substituting $\omega_u = \dfrac{\omega_u' e}{w_u' e} \cdot w_u$ $(u = 1, \cdots, k)$, in (A.7) and (A.11). A necessary condition for mean squared error to be zero is that $\omega^{kl} = w$, which implies $w_u'/w_u' e = \omega_u'/\omega_u' e$ $(u = 1, \cdots, k)$.

This condition for perfect accuracy requires only that the ratio of index to firm weights be the same for all goods included in an index. If goods 1 and 2 are included in the same index, ω_1/w_1 must be equal to ω_2/w_2; if they are in separate indexes, the condition is not necessary for perfect accuracy. This weaker condition implies that for any exchange valuation rule, R_{kl}, there is a set of firms S_{kl} for which it has perfect accuracy. This set is convex in the sense that if two firms belong to this set, their linear combinations also belong to this set and therefore valuation rule R_{kl} has a mean squared error of zero for them. Formal statement of this result, Proposition 4, and one of its corollaries are proved in the Appendix A.

The notion that the use of specific price indexes results in more accurate estimation of current value than a single-index valuation seems to be fairly widespread. The main result of this section is that, if the statistical measures of accuracy used here capture that notion of accuracy, the assumption does not hold for individual firms.

4. Average Accuracy of Exchange Valuation Systems.

The preceding analysis for individual firms indicates that the accuracy of exchange valuation systems is not a monotonic function of the level of disaggregation of the price indexes used. Of any pair of exchange valuation systems, one may be more accurate for some firms and the other for the rest. If the valuation rule is regarded as a matter of social choice, and if the accuracy of valuation of individual firms is to be the criterion, it is clear that valuation rules cannot be ranked with respect to accuracy if the high-ranking rule is required to be at least as accurate as the lower-ranking rules for every firm. In the presence of this difficulty, we must search for other criteria that could be used to rank the valuation rules with respect to their accuracy.

A cross-sectional average of the accuracy of individual firms is a reasonable alternative criterion. Under this criterion, we shall assign a higher rank to a valuation rule which, on average, has greater accuracy than another, even if this higher-ranking rule may be less accurate for some specific firms. The criterion of average accuracy is obviously weaker than than the criterion of accuracy for individual firms. It does, however, provide a complete ranking of all exchange valuation rules,[14] as we shall show in the following.

In order to derive the expressions for average accuracy of exchange valuation systems, the following assumptions are made: (1) The economy consists of N firms of equal size. (2) The relative weights used for construction of price indexes are the relative weights of the goods in the entire economy. (3) The asset portfolio of each firm can be considered as

[14] The relationship between the two criteria, specific firm accuracy and average accuracy of valuation rules for the purpose of ranking the rules, is roughly analogous to the relationship between Pareto's [1927] and Kaldor's [1939] potential Pareto criteria for ranking the wealth distributions in the society, another social choice problem. Note that the usual application of Pareto and potential Pareto criteria is found in problems of social choice arising out of interpersonal differences in utility; the analogy drawn here is to a social choice problem with differences of social utility with respect to various firms.

The potential Pareto principle involves the concept of a hypothetical costless transfer of wealth which has no analogue in this case because it is not meaningful to talk about the transfer of accuracy from one firm to another. The analogy is limited but useful. The compensation principle implies that the interpersonal average of wealth or utility can be used as the social criterion; the present extension implies that the interfirm average of accuracy can be used as the social criterion. In the absence of maximal alternatives, this seems to be the best approach to analysis.

a multinomial random vector drawn from the asset pool of the entire economy using ρ trials. Subject to these assumptions, the cross-sectional expectation of valuation bias and mean squared error associated with various exchange valuation rules has been derived in Appendix A. The results are summarized in columns 5 and 6 of table 1.

Average Bias. The average bias for all exchange valuation rules except one is zero. Only the historical cost valuation has a downward (upward) bias when the expected change in price-level is positive (negative). Though the k-index ($k = 1, 2, \cdots, n - 1$) valuation rules are not unbiased for individual firms, the cross-sectional average of their bias is zero for each one. Therefore, the average bias cannot help discriminate among the elements of an exchange valuation set with the sole exception of historical cost.

Average Mean Squared Error (AMSE). I shall state the main findings of the investigation in the form of two propositions.

Proposition 5. The average mean squared error of a k_1-index valuation system using the index configuration π_{k_1i} is less than the *AMSE* of a k_2-index valuation system using an index configuration π_{k_2j}, provided that π_{k_1i} is strictly finer than π_{k_2j}. In other words, a finer index system results in a lower *AMSE*. Proof of the proposition is given in Appendix A. Since the single- and n-index systems are, respectively, the coarsest and the finest of all, it immediately follows that the *AMSE* of any k-index valuation system ($1 < k < n$) is less than the *AMSE* of general price level valuation, but still greater than the *AMSE* of the current valuation. The latter quantity is zero by definition.

A comparison between the *AMSE*'s of single-index valuation and historical valuation indicated that the latter is always larger by $\omega'(\sum + \mu\mu')\omega$, a nonnegative quantity. This allows us to extend Proposition 5 to include historical valuation. The average accuracy of single- and all multi-index valuation systems exceeds that of historical valuation; the finer the index configuration, the lower is the *AMSE*. Therefore, if *AMSE* were to be used as a measure of accuracy, and accuracy were to be one of the criteria for selecting a valuation system, we could select one of a pair of exchange valuation systems simply by comparing the fineness of the associated index configurations and without having to calculate the respective *AMSE*'s.

But all index configurations, that is, partitions of a set G of n goods, are not comparable with respect to their fineness. Fineness induces only a partial ordering on partitions of G, because it is possible for a partition to be neither finer nor coarser than another. The number of price indexes, that is, the number of partitions k, does induce a complete, though weak, ordering on the index valuation systems because the number of price indexes used in one system must be less than, equal to, or greater than the number used in another system. It would be very convenient if the accuracy of two exchange valuation systems could be compared simply by looking at the number of indexes used. The notion that the larger the number of price indexes used, the greater the accuracy of valuation, has considerable intuitive appeal. Unfortunately, *AMSE*, like *MSE* for individual firms, is not a monotonic function of the number of price indexes used. It takes only a small numerical example to prove the following proposition (see Appendix A).

Proposition 6. The economy-wide average of the mean squared error associated with an exchange valuation does not necessarily decrease as the number of price indexes used is increased. To fix ideas, I have calculated the *AMSE* for the simple example used to prove Proposition 6. The results are given in the last columns of table 3 and in figure 2.

There are seven and six ways of forming two and three indexes, respectively, for four goods. There is only one way each of forming one and four indexes. Including the historical valuation, there are sixteen elements in the exchange valuation set. In figure 2, each node represents an exchange valuation rule and its average mean squared error

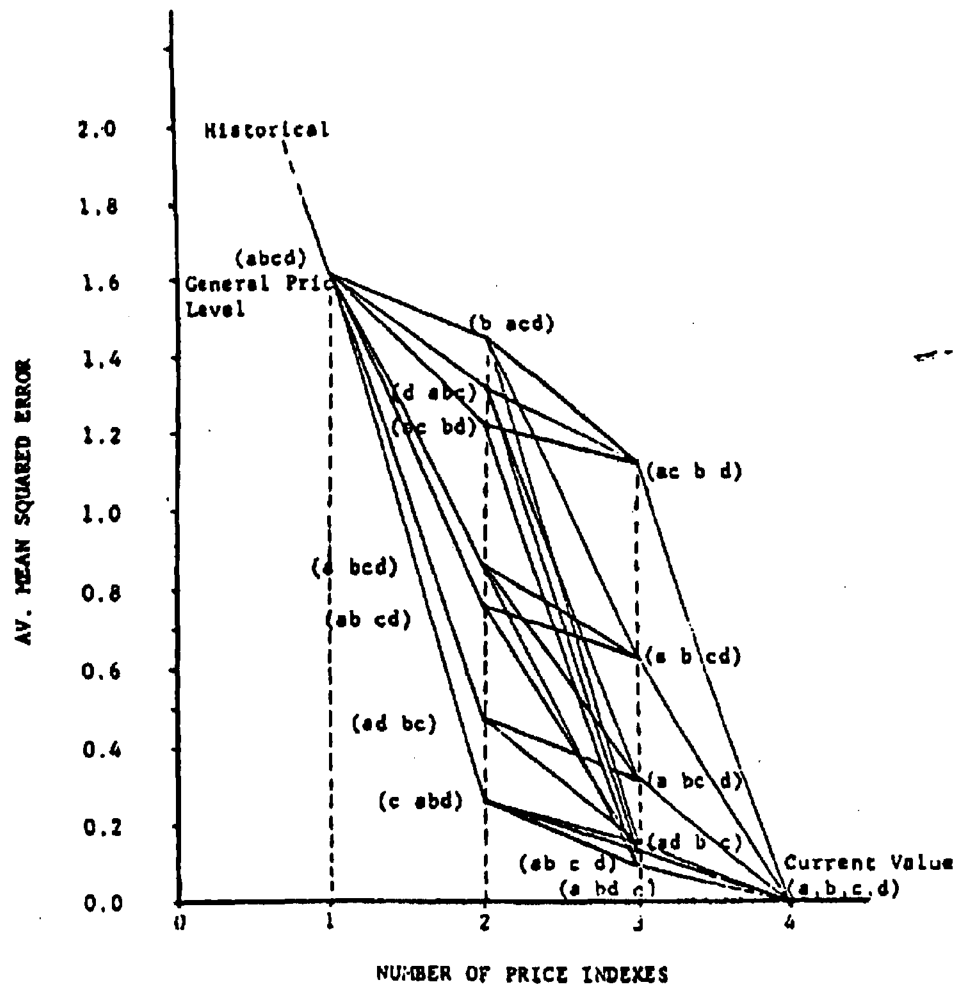

FIG. 2.—Average mean squared error of exchange valuation rules. Letters in parentheses indicate the price index configurations for four goods, G (a, b, c, d). Valuation systems comparable with respect to fineness are joined by straight lines.

has been plotted against the number of indexes. As in figure 1, each node has been connected by straight lines to strictly coarser index configurations to the left and to strictly finer index configurations to the right. It is immediately seen that the *AMSE* is not a decreasing function of the number of indexes. As we move from left to right along any of the lines joining the nodes which have a strictly coarser-finer relationship, we encounter progressively finer index configurations. In accordance with Proposition 5, the *AMSE* of strictly finer index valuation systems is strictly smaller.

It is interesting to compare figure 2 with figure 1, in which the *MSE* of the same valuation systems has been plotted for an individual firm whose vector of relative proportions of four goods is (0.1, 0.3, 0.15, 0.45). Again, strictly finer-coarser systems are connected by straight lines, but these lines do not necessarily have a negative slope, since the mean square error for a specific firm has no direct relationship with the number or fineness of price indexes used.

Figure 2 is only a small example of how the exchange valuation rules can be directly compared in terms of their quantified attributes—in this instance the accuracy. I hope that subsequent research will make it possible to plot these systems along other dimensions on the basis of attributes of verifiability and cost. Visualizing the valuation alternatives in a three-dimensional space will help to focus the discussion on further refinements in measuring the relevant attributes and social trade-offs among them.

To summarize the above findings, it has been shown that (a) the bias and mean squared error of valuation of individual firms are not monotonic functions of either the number or the fineness of price indexes used; (b) the cross-sectional average of valuation bias is zero for all exchange valuation systems except the historical one; (c) the average *MSE* of valuation does not necessarily decrease with the increasing number of price indexes; and (d) it does always decrease with increasing fineness of price indexes. In addition, we have developed expressions to calculate the bias and *MSE* of all possible exchange valuation rules. *MSE*, as well as *AMSE*, does induce a complete ordering on the partitions of sets of goods G, and it is always possible to determine which of a pair of exchange valuation rules is more desirable in terms of accuracy. Since the total number of exchange valuation systems is very large even for moderate values of n, a search for the system with the lowest *AMSE* by this procedure is likely to be very laborious. The development of an efficient algorithm for identifying the most accurate of the k-index systems for $k = 2, 3, \ldots, n - 1$ would be a fruitful area for further work.

5. Other Attributes of Exchange Valuation Rules and the Choice Problem

In the preceding sections, I have defined the analytical structure of the family of exchange valuation rules and analyzed their accuracy in approximating the current value. While accuracy is an important characteristic of valuation rules, it is not the only one relevant to the problem of choice of valuation rules. In this section, I briefly discuss two other important characteristics of valuation rules, objectivity and cost, which can be analyzed with the help of the algebraic structure already developed.

Objectivity. Many accounting procedures are not specified in completely operational terms, in the sense that they allow or require the accountant to use his judgment and discretion. Exchange valuation rules, too, allow varying amounts of discretion to their user, giving rise to differences in valuation arrived at by different individuals under the same rule. Other things being equal, a valuation rule which results in smaller cross-sectional differences is regarded as more desirable and is referred to as being more objective.

The relationship among the exchange valuation rules with respect to their objectivity can be analyzed with the help of the analytical structure of the exchange valuation set. The first step in the application of an exchange valuation rule requires that all assets of the firm be classified into an appropriate number of groups. Since exchange valuation rules differ with respect to the nature and extent of classificatory judgments required of the managers/accountants in their application, they also differ in their objectivity. The differences in objectivity in terms of algebraic structure of valuation rules remain to be characterized.

Cost. The possibility of being able to convert the accuracy and objectivity attributes of valuation rules into monetary terms is remote at best. However, there is a better chance that certain cash consequences of using alternative valuation rules can be identified, and the term cost is used here to denote these consequences. This includes the cost of data collection, computation, and internal and external audits. The cost of conversion from the prevailing to the new system must also be considered. Some would argue that if one valuation rule leads to more litigation than others, the cost of such litigation should be included in making cost comparisons. Some preliminary cost data on single- and multi-index valuations are already available (Bell [1971, p. 31]), though much work remains to be done.

The social choice problem could be considered in three stages—definition of the socially desirable or undesirable attributes of valuation systems, measurement of these attributes, and comparison among the valuation systems to make a choice. I deal with the first, and part of the second, problem by adopting the three attributes of accuracy, objectivity, and cost, whose social relevance might be broadly acceptable and by proposing a method of measuring accuracy. I have no illusion of there being unanimous agreement even on these attributes. I am not aware, however, of other attributes of valuation systems which have much broader acceptance.

Relevance to decisions has frequently been mentioned as a desirable attribute of valuation rules, but considerable difficulty is encountered when attempts are made to define relevance in a generally acceptable way. Part of the reason is given by Vatter [1971, 130–31]: ". . . decision-making must be based on relevant information. But relevance is not an inherent characteristic of data, nor a way of viewing markets or measurement methods. Relevance is the degree to which data reflect the relations that exist (or ought to exist) between problems and their resolution in the decision process."

Since relevance is a problem-specific attribute of valuation, each valuation procedure will have to be evaluated in terms of relevance of the data produced to different types of problems. Under conditions of perfect markets, the usefulness of current-value data to decisions is readily granted, and it is parsimonious to evaluate the accuracy with which various valuation rules surrogate the current values. Accuracy is therefore a measure of relevance. Freedom from bias, often mentioned as an important attribute of valuation (American Accounting Association [1966]), can similarly be subsumed under the term accuracy, as was seen in Section 3.

The choice of a valuation system can be viewed as a problem of making a choice among objects with multiple, noncomparable attributes. There is little in the way of a formal theory of such choices, and it can only be hoped that a clearer, more precise measurement of the attributes will facilitate comparative examination of valuation rules, or at least help to focus the discussion on this last problem by clearly presenting the choice alternatives within a unified framework.

6. *Concluding Remarks*

I have defined a general set of valuation rules based on exchange prices in an attempt to develop a unified theory of valuation which allows various existing systems of accounting valuation to be viewed as special cases of the general exchange valuation set. This approach to the study of the theories of valuation offers a distinct advantage in that it facilitates quantitative comparison between the attributes of valuation rules. Such quantification of differences between valuation rules as contrasted to the nominal classification of the systems (e.g., historical versus general purchasing power versus current cost) will, I believe, prove to be more useful in their systematic comparison.

It is appropriate to note that the analytical framework developed here does not encompass all elements of exchange valuation at once. We have divided the set into two equal and isomorphic subsets for entry and exit price systems. The analysis can be used to compare any valuation system with any other within, but not between, the subsets. Comparisons between the subsets will need a more elaborate modeling of the asset market imperfections which give rise to differences between entry and exit prices.

In the present study, one attribute of the exchange valuation rules—

the accuracy of valuation with the bias and mean squared error taken as the measures of accuracy—has been examined. I have derived relationships which exist between the accuracy of various elements of the exchange valuation set when applied to an individual firm and to all firms in an economy. Companion studies are planned to make similar comparisons of other relevant attributes of valuation rules. A systematic study of accuracy, objectivity, and cost of valuation rules will, I hope, provide a sound basis for comparative analysis of accounting valuation systems.

APPENDIX A

Consider an ordered set G of n distinct goods.

Definition 1. Let G be a set of n elements and π_{kl} be a set of k nonempty subsets of G. π_{kl} is a *partition* of G if and only if all subsets in π_{kl} are mutually exclusive and their union is G.

Definition 2. Partition $\pi_{k_1 i}$ of G is finer than partition $\pi_{k_2 j}$ of G if and only if each subset in $\pi_{k_1 i}$ is included in some subset in $\pi_{k_2 j}$. Then $\pi_{k_2 j}$ is coarser than $\pi_{k_1 i}$.

Definition 3. Partition $\pi_{k_1 i}$ of G is *strictly* finer than partition $\pi_{k_2 j}$ of G if and only if it is finer than $\pi_{k_2 j}$ but is not equal to $\pi_{k_2 j}$.

Relative price changes for the goods from time 0 to 1 are given by elements of the $(n \times 1)$ vector $\mathbf{r}$. Let ω and $\mathbf{w}$ be $(n \times 1)$ vectors of the relative weights of the general price index and the basket of goods being evaluated, respectively. The principal aggregation is $R^* = \mathbf{w}'\mathbf{r}$. Historical, single-index, n-index, and k-index ($k = 1, \ldots, n$) estimators of the relative change in price of the basket of goods $\mathbf{w}$ have been defined in Section 1.

$$R_{0,1} = 0. \tag{A.1}$$

$$R_{1,1} = \omega'\mathbf{r}. \tag{A.2}$$

$$R_{n,1} = \mathbf{w}'\mathbf{r} = R^*. \tag{A.3}$$

To write the general case of the k-index estimator, let L_k be the total number of k-partitions of set G. Let π_{kl} ($i = 1, 2, \ldots, L_k$) be the ith of these partitions and R_{kl} be the estimator of the relative price change obtained from the index configuration corresponding to π_{kl}. Without loss of generality, let:

$$\omega' = (\omega_1' \mid \omega_2' \mid \cdots \mid \omega_k')$$
$$\mathbf{w}' = (\mathbf{w}_1' \mid \mathbf{w}_2' \mid \cdots \mid \mathbf{w}_k')$$

and

$$\mathbf{r}' = (\mathbf{r}_1' \mid \mathbf{r}_2' \mid \cdots \mid \mathbf{r}_k')$$

be the corresponding partitions of ω, $\mathbf{w}$, and $\mathbf{r}$ respectively. Then R_{kl} can be written as:

$$R_{kl} = \sum_{v=1}^{k} \left(\frac{\mathbf{w}_v'\mathbf{e}}{\omega_v'\mathbf{e}} \, \omega_v'\mathbf{r}_v \right). \tag{A.4}$$

Since the choice between valuation systems is to be made before the financial statements are prepared, their accuracy in surrogating the principal quantity also must be evaluated on an ex ante basis. At time zero, the prospective relative price changes r between time 0 and 1 have a probability distribution which, for the present purposes, can be sufficiently described by the mean vector μ and covariance matrix Σ. The accuracy of valuation systems can be evaluated using the analytical definitions of the valuation systems already given and the mean and variance of prospective price changes. A tilde is added to R^* and r to indicate that, ex ante, they are considered random variables.

BIAS

Historical $$E(R_{0,1} - \tilde{R}^*) = E(-\mathbf{w}'\tilde{\mathbf{r}}) = -\mathbf{w}'\mu. \tag{A.5}$$

Single-Index $$E(R_{1,1} - \tilde{R}^*) = E(\omega'\tilde{\mathbf{r}} - \mathbf{w}'\tilde{\mathbf{r}}) = (\omega - \mathbf{w})'\mu. \tag{A.6}$$

k-Index $$E(R_{kl} - \tilde{R}^*) = E\sum_{u=1}^{k}\left(\frac{\mathbf{w}_u'\mathbf{e}}{\omega_u'\mathbf{e}}\,\omega_u'\tilde{\mathbf{r}}_u\right) = (\omega^{kl} - \mathbf{w})'\mu \tag{A.7}$$

where

$$\omega^{kl} = \left(\frac{\mathbf{w}_1'\mathbf{e}}{\omega_1'\mathbf{e}}\,\omega_1' \,\middle|\, \frac{\mathbf{w}_2'\mathbf{e}}{\omega_2'\mathbf{e}}\,\omega_2' \,\middle|\, \cdots \,\middle|\, \frac{\mathbf{w}_k'\mathbf{e}}{\omega_k'\mathbf{e}}\,\omega_k'\right)'.$$

n-Index $$E(R_{n,1} - \tilde{R}^*) = E(\mathbf{w}'\tilde{\mathbf{r}} - \mathbf{w}'\tilde{\mathbf{r}}) = 0. \tag{A.8}$$

MEAN SQUARED ERROR

Historical $$E(R_{0,1} - \tilde{R}^*)^2 = \mathbf{w}'(\Sigma + \mu\mu')\mathbf{w}. \tag{A.9}$$

Single-Index $$E(R_{1,1} - \tilde{R}^*)^2 = (\omega - \mathbf{w})'(\Sigma + \mu\mu')(\omega - \mathbf{w}). \tag{A.10}$$

k Index $$E(R_{kl} - \tilde{R}^*)^2 = (\omega^{kl} - \mathbf{w})(\Sigma + \mu\mu')(\omega^{kl} - \mathbf{w}). \tag{A.11}$$

n-Index $$E(R_{n,1} - \tilde{R}^*)^2 = 0. \tag{A.12}$$

PROOF OF PROPOSITION 4

Proposition 4. For every exchange valuation rule R_{kl} ($k = 1, \ldots, n$; $i = 1, \ldots, L_k$), there exists a convex nonempty set $S_{kl} = \{\mathbf{w}\}$ of firms for which valuation rule R_{kl} has a mean squared error of zero. If $(\omega_1' \mid \omega' \mid \cdots \mid \omega_k')'$ is the partition of ω corresponding to R_{kl}, set S_{kl} is given by:

$$\left\{\mathbf{w} \middle| \mathbf{w} = \left(\frac{\lambda_1}{\omega_1'\mathbf{e}}\,\omega_1' \,\middle|\, \frac{\lambda_2}{\omega_2'\mathbf{e}}\omega_2' \,\middle|\, \cdots \,\middle|\, \frac{\lambda_k}{\omega_k'\mathbf{e}}\omega_k'\right);\ \lambda_u \geq 0,\ u = 1, \cdots, k;\ \lambda_1 + \lambda_2 + \cdots + \lambda_k = 1\right\}.$$

Substitution of $\mathbf{w} = \left(\frac{\lambda_1}{\omega_1'\mathbf{e}} \mid \cdots \mid \frac{\lambda_k}{\omega_k'\mathbf{e}}\,\omega_k'\right)'$ in (A.11) is sufficient to prove that every firm in S_{kl} has zero mean squared error with respect to valuation rule R_{kl}. To confirm that all firms for which $MSE\,(R_{kl})$ is zero are included in S_{kl}, note that a necessary and sufficient condition for inclusion is $\mathbf{w}_u = \frac{\mathbf{w}'\mathbf{e}}{\omega_u'\mathbf{e}}\cdot\omega_u$ for $u = 1, \ldots, k$. Since the only restrictions on $\mathbf{w}_u'\mathbf{e}$ are that it be nonnegative and $\sum_{u=1}^{k}\mathbf{w}_u'\mathbf{e} = 1$, we can substitute $\mathbf{w}_u'\mathbf{e} = \lambda_u$, $\lambda_u \geq 0$, $\sum_{u=1}^{k}\lambda_u = 1$, which yields the definition of set S_{kl}. The set is nonempty because $\mathbf{w} = \omega$ is obviously a member. To prove convexity of set S_{kl}, suppose that firms $\hat{\mathbf{w}}$ and $\hat{\hat{\mathbf{w}}}$ are members of the set. Let $\overset{*}{\mathbf{w}}$ be a convex combination of these two firms, i.e., $\overset{*}{\mathbf{w}} = \alpha\,\hat{\mathbf{w}} + (1$

$- \alpha)\dot{\hat{w}}$, $(0 \leq \alpha \leq 1)$. For $u = 1, 2, \ldots, k$, $\hat{w}_u = \frac{\hat{w}_u'e}{\omega_u'e}\omega_u$ and $\dot{\hat{w}}_u = \frac{\dot{\hat{w}}_u'e}{\omega_u'e}\omega_u$; therefore $\overset{*}{w}_u = \alpha\hat{w}_u + (1 - \alpha)\dot{\hat{w}}_u = \alpha\frac{\hat{w}_u'e}{\omega_u'e}\omega_u + (1 - \alpha)\frac{\dot{\hat{w}}_u'e}{\omega_u'e}\omega_u$ or $\overset{*}{w}_u = \frac{\overset{*}{w}_u'e}{\omega_u'e}\omega_u$ for $u = 1, 2, \ldots, k$, which is both necessary and sufficient for $\overset{*}{w}$ to be an element of S_{kl}. A corollary of Proposition 4 is the following.

Proposition 4.1. If the mean squared error of an exchange valuation rule R_{kl} is zero with respect to a given firm w, the mean squared error of all exchange valuation rules finer than R_{kl} for the firm is also zero. The proof of the proposition is immediate from the necessary and sufficient condition $w_u = \frac{w'_u e}{\omega_u'e}\omega_u$, $u = 1, 2, \ldots, k$ for zero mean squared error. In a finer valuation rule, one or more of the price indexes used in R_{kl} must be further partitioned and the condition will obviously continue to hold for each index.

AVERAGE ACCURACY

Under the assumption that all N firms are equal in size, the economy-wide mean of vectors $w(j)$ is, by definition, equal to ω. Therefore:

$$\frac{1}{N}\sum_{j=1}^{N} w(j) = \omega \tag{A.13}$$

where both $w(j)$ and ω are vectors of proportions. $w(j)$ can therefore be viewed as a random variable derived from multinomial distribution with mean vector of proportions ω and a parameter ρ representing the number of trials used to draw the multinomial sample.[15] The mean vector and covariance matrix of random variable $\tilde{w}(j)$ (to be written as $\tilde{w}$ for simplicity) are:

$$E(\tilde{w}) = \omega \tag{A.14}$$

$$\mathrm{Var}(\tilde{w}) = \Delta = \frac{1}{\rho}\begin{bmatrix} \omega_1(1-\omega_1) & -\omega_2\omega_1 & \ldots & -\omega_n\omega_1 \\ -\omega_1\omega_2 & \omega_2(1-\omega_2) & \ldots & -\omega_n\omega_2 \\ -\omega_1\omega_3 & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & & & \\ -\omega_1\omega_n & -\omega_2\omega_n & \ldots & \omega_n(1-\omega_n) \end{bmatrix} \tag{A.15}$$

or

$$\delta_{ik} = \mathrm{Cov}(\tilde{w}_i, \tilde{w}_k) = \begin{cases} \frac{1}{\rho}\omega_i(1-\omega_i) & \text{if } i = k \\ -\frac{1}{\rho}\omega_i\omega_k & \text{if } i \neq k \end{cases} \tag{A.16}$$

[15] A discussion I had with Stan Garstka was very useful in modeling this distribution. I alone am responsible for the possible errors in this application.

and

$$E(\dot{w}_i \cdot \dot{w}_k) = \begin{cases} \dfrac{\rho - 1}{\rho}\omega_i^2 + \dfrac{\omega_i}{\rho} & \text{if } i = k \\ \dfrac{\rho - 1}{\rho}\cdot\omega_i\omega_k & \text{if } i \neq k \end{cases} \tag{A.17}$$

From this cross-sectional distribution of relative weights, we can derive expressions for the economy-wide averages of bias (AB) and mean squared error ($AMSE$) associated with various estimators.

AVERAGE BIAS

Historical $$E_{\dot{w}}\{E_t(R_{0,t}) - \dot{R}^*)\} = -\omega'\mu. \tag{A.18}$$

Single-Index $$E_{\dot{w}}\{E_t(R_{1,t}) - \dot{R}^*)\} = 0. \tag{A.19}$$

k-Index $$E_{\dot{w}}\{E_t(R_{kt} - \dot{R}^*)\} = 0. \tag{A.20}$$

n-Index $$E_{\dot{w}}\{E_t(R_{n,t} - \dot{R}^*)\} = 0. \tag{A.21}$$

AVERAGE MEAN SQUARED ERROR

Historical $$E_{\dot{w}}\{E_t(R_{0,t} - \dot{R}^*)^2 = E_{\dot{w}}\{\dot{w}'(\textstyle\sum + \mu\mu')\dot{w}\} \tag{A.22}$$

$$= \frac{1}{\rho} \quad \{\omega'(\sigma + \dot{\mu}) - \omega'(\textstyle\sum + \mu\mu')\omega\} + \omega'(\textstyle\sum + \mu\mu')\omega$$

where σ is the vector of diagonal elements of $\sum$, and $\dot{\mu}$ is obtained from μ by squaring each element, $\dot{\mu}_i = \mu_i^2$ for $i = 1, 2, \ldots, n$.

Single-index $E_{\dot{w}}\{E_t(R_{1,t} - \dot{R}^*)^2 = E_{\dot{w}}\{(\omega - \dot{w})'$

$$\cdot(\textstyle\sum + \mu\mu')(\omega - \dot{w})\} = \frac{1}{\rho}\{\omega'(\sigma + \dot{\mu}) - \omega'(\textstyle\sum + \mu\mu')\omega\}. \tag{A.23}$$

k-Index $E_{\dot{w}}\{E_t(R_{kt}) - \dot{R}^*)^2\} = E_{\dot{w}}\{(\omega^{kt} - \dot{w})(\sum + \mu\mu')(\omega^{kt} - \dot{w})\}$

$$= \frac{1}{\rho}\{\omega'(\sigma + \dot{\mu}) - \sum_{u=1}^{k}(\omega_u'(\textstyle\sum_{uu} + \mu_u\mu_u')\,\omega_u/\omega_u'e)\} \tag{A.24}$$

where $\sum_{uu}$ is the submatrix of $\sum$ corresponding to the uth subset in π_{kt}.

n-Index $$E_{\dot{w}}\{E_t(R_{n,t} - \dot{R}^*)^2\} = 0. \tag{A.25}$$

PROOF OF PROPOSITION 5

Proposition 5. If partition π_{k_1l} of G is strictly finer than partition π_{k_2J}, the economy-wide average of the mean squared error of R_{k_1l} is less than the corresponding error of R_{k_2J}, that is:

$$\pi_{k_1l} \subset \pi_{k_2J} \Longrightarrow AMSE(R_{k_1l}) < AMSE(R_{k_2J}). \tag{A.26}$$

Proof. By definition, π_{k_1l} has k_1 subsets and π_{k_2J} has k_2 subsets. Since π_{k_1l} is strictly finer than π_{k_2J}, k_1 is strictly greater than k_2; i.e., the number of subsets in π_{k_1l} exceeds the number of subsets in π_{k_2J} by at least 1. Let $k_2 = m$ and without loss of generality, let $k_1 = m + 1$. Since there is one more subset in $\pi_{m+1,l}$ than in π_{mJ} and since $\pi_{m+1,l} \subset \pi_{mJ}$ also assume without loss of generality that $\pi_{m+1,l}$ is obtained from π_{mj} by subdividing the last subset of π_{mj} which has ν elements into two subsets consisting of first ν_1 and last $\nu_2 = \nu - \nu_1$ elements, respectively. This arrangement implies that the first $(m - 1)$ subsets in π_{mj} and $\pi_{m+1,l}$ are identical. The last two subsets in $\pi_{m+1,l}$ can be combined to obtain the last subset in π_{mj}.

From (A.24)

$$AMSE\ R_{mj} = \frac{1}{\rho}\left\{\omega'(\sigma + \dot{\mu}) - \sum_{u=1}^{m} \frac{1}{\omega_u'e}\,\omega_u'\left(\textstyle\sum_{uu} + \mu_u\mu_u'\right)\omega_u\right\} \quad (A.27)$$

$$AMSE(R_{m+1,l}) = \frac{1}{\rho}\left\{\omega'(\sigma + \dot{\mu}) - \sum_{r=1}^{m+1} \frac{1}{\omega_r'e}\,\omega_r'\left(\textstyle\sum_{rr} + \mu_r\,\mu_r'\right)\omega_r\right\} \quad (A.28)$$

where subscript u is used to index m subsets of G according to partition π_{mj} and ν for $(m + 1)$ subsets according to partition $\pi_{m+1,l}$. Since the first $(m - 1)$ subsets in π_{mj} and $\pi_{m+1,l}$ are identical:

$$\begin{aligned} AMSE(R_{mj}) - AMSE(R_{m+1,l}) = -\frac{1}{\rho}\Bigg\{&\sum_{u=m}^{m} \frac{1}{\omega_u'e}\,\omega_u'\left(\textstyle\sum_{uu} + \mu_u\mu_u'\right)\omega_u \\ &- \sum_{r=m}^{m+1} \frac{1}{\omega_r'e}\,\omega_r'\left(\textstyle\sum_{rr} + \mu_r\mu_r'\right)\omega_r\Bigg\}. \end{aligned}$$

if ν members of the mth subset in π_{mj} are indexed by $s, t = 1, 2, \ldots, \nu$.

$$\begin{aligned} = \frac{1}{\rho}\Bigg[-\sum_{s=1}^{\nu}\sum_{t=1}^{\nu} \frac{(\sigma_{st} + \mu_s\mu_t)\omega_s\omega_t}{\sum_{s=1}^{\nu} \omega_s} + \sum_{s=1}^{\nu_1}\sum_{t=1}^{\nu_1} \frac{(\sigma_{st} + \mu_s\mu_t)\omega_s\omega_t}{\sum_{s=1}^{\nu_1} \omega_s} \\ + \sum_{s=\nu_1+1}^{\nu}\ \sum_{t=\nu_1+1}^{\nu} \frac{(\sigma_{st} + \mu_s\mu_t)\omega_s\omega_t}{\sum_{s=\nu_1+1}^{\nu} \omega_s}\Bigg] \end{aligned}$$

$$= \frac{1}{\rho}\sum_{s=1}^{\nu}\sum_{t=1}^{\nu}(\sigma_{st} + \mu_s\mu_t)d_sd_t$$

$$\text{where } d_s = \begin{cases} \omega_s\sqrt{\dfrac{\sum_{t=\nu_1+1}^{\nu} \omega_t}{\sum_{t=1}^{\nu} \omega_t \cdot \sum_{t=1}^{\nu_1} \omega_t}} & \text{if } 1 \le s \le \nu_1 \\[3ex] -\omega_s\sqrt{\dfrac{\sum_{t=1}^{\nu_1} \omega_t}{\sum_{t=\nu_1+1}^{\nu} \omega_t \sum_{t=1}^{\nu} \omega_t}} & \text{if } \nu_1 < s \le \nu \end{cases}$$

$$= \frac{1}{\rho}\,\mathrm{d}\,(\textstyle\sum + \mu\mu')\mathrm{d} > 0 \quad (A.29)$$

since $(\sum + \mu\mu')$ is positive definite.

PROOF OF PROPOSITION 6

Proposition 6. A larger value of k does not necessarily result in a lower economy-wide average of mean squared error of estimator R_{kt}, that is:

$$k_1 > k_2 \mp > AMSE(R_{k_1 l}) < AMSE(R_{k_2 J}). \tag{A.30}$$

Proof. To prove the theorem, it is enough to give a counterexample. Consider the following parameters for a set $G = (a, b, c, d)$ with $n = 4$ goods.

$$\omega' = (0.3, 0.05, 0.5, 0.15).$$

$$\mu = 0.$$

$$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}.$$

Let $k_2 = 2$, consider partition $\pi_{2,J}$ of G given by $((c), (abd))$. The corresponding value of $AMSE(\pi_{2,J})$ is calculated from (A.24):

$$AMSE(R_{2,J}) = \frac{0.2625}{\rho}.$$

For $k_1 = 3 > k_2 = 2$, consider partition $\pi_{3,l}$ of ω given by $((a), (b), (cd))$. The corresponding average of mean squared error is:

$$AMSE(R_{3,l}) = \frac{0.6346}{\rho} AMSE(R_{2,J}).$$

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