A MULTI-PERIOD INTEGER PROGRAMMING APPROACH TO THE PRODUCT MIX PROBLEM

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ABSTRACT

This paper presents an integer programming approach to the product mix problem. The model considers revenue interactions among products and permits the addition and deletion of products over a multi-period planning horizon. The paper also discusses parameter estimation requirements, and variations of the model's application to the product mix decision problem.

INTRODUCTION

The problem of determining an optimal product mix is an important problem faced by multi-product firms. Product mix decisions involve introduction of new products into the market and continuation or withdrawal from the market of the current products. In the absence of cost and demand interdependencies, introduction or withdrawal of each product is an independent decision to be made on its own merits. However, frequent presence of interdependence among products implies that the firm must determine its entire product mix to maximize its profits. Fragmented decisions involving consideration of one product at a time would result in a suboptimal product mix.

The product mix problem is essentially dynamic. The problem is not one of selecting a subset of products at time $\underline{t}$ and holding them for a given period of time, but, rather of adding and dropping products over a planning horizon. One aspect of the problem is to recognize different products will have profitable life cycles that differ in length and in the discounted value of the earnings stream. Thus, over the planning horizon the composition of the product mix should change as products are added or deleted consistent with the firm's profit objective.

The literature has reported only two previous attempts to solve the above product mix problem. Rice [7] developed a methodology for selecting a product line from a set of available lines on the basis of maximum expected net contribution. He did not consider the interdependencies which represent the essence of the product mix problem. Nor did his model allow dynamic adjustment of product mix as more information becomes available in succeeding periods. Using a capital investment framework Kotler [4] considered the dynamic planning and product interaction aspects of the product mix problem. However, he presented only a graphical illustration and suggested that there are no mathematical programming algorithms for selecting the best solution [4,p.185].

This paper presents an integer programming approach to the product mix problem. Interaction effects between revenues earned by the various products are explicitly considered. Product introduction and deletion decisions are simultaneously determined to maximize the firm's objective function over the planning horizon subject to specific resource constraints. An example is presented to illustrate the computer feasibility of the approach.

THE MODEL

Model Parameters and Input Requirements

Assume that a firm has a planning horizon of $\underline{T}$ periods, and has $\underline{m}$ existing products on the market at the beginning of period 1. The firm also has $\underline{n}$ new products available for introduction that can be introduced to the market in any period from 1 to T.

In the following definitions, subscript i for $1\leq i\leq m$ denotes the existing products and for $m+1\leq i\leq m+n$ denotes the new products.

<u>Product life cycles</u>.

$C_{i,t}$ is the estimated total cost of producing and marketing product i in period t after its introduction for i=1, ...,m,...,m+n and t=1,...,T, treating the existing products as if they were introduced at the beginning of period 1.

$L_{i,t}$ is the estimated revenue from product i in period t after its introduction for i=1,...,m...,m+n and t=1, ...,T.

<u>Product interaction</u>.

$d_{i,j,t}$ is the fraction of estimated revenue, $L_{i,t}$, representing the interaction effect on the revenue of product i in period t after introduction of product i due to its coexistence in the market with product j for i=1,...,m+n; j=1,...,m+n; i≠j; t=1,...,T.

Decision Variables

<u>Product introduction variables</u>.

$X_{j,t}$ = 1 if product (m+j) is introduced in period t, j=1, ...,n and t=1,...,T.
0 otherwise

<u>Product withdrawal variables</u>.

$Y_{j,t}$ = 1 if product j is withdrawn at beginning of period t, j=1,...,m and t=1,...,T.
0 otherwise

Interaction Variables

$Z_{i,j,t}$ = 1 if the ith and jth products are simultaneously on the market in the tth period, i=2,...,m+n; j=1,...,i-1; t=1,...,T.
0 otherwise

$W_{i,j,t,u}$ = 1 if the ith and jth products are simultaneously on the market in period t and product i was introduced in period u; i=m+1,...,m+n; j=1,...,m+n; i≠j; t=1,...,T; u=1,...,t.
0 otherwise

Objective Function

Assume the firm's objective is to maximize discounted present value of net cash flows via product introductions and withdrawals over the T periods. Let the discounting factor be a α per period. Then the objective is

$$\text{Max } [PV_1+PV_2+PV_3+PV_4+PV_5] \qquad (1)$$

where

PV_1 is present value of net cash flows from existing products, without interactions

$$= \sum_{i=1}^{m}\sum_{t=1}^{T}\alpha^{t-1}(L_{i,t}-C_{i,t}) - \sum_{i=1}^{m}\sum_{t=1}^{T}Y_{i,t}\sum_{s=t}^{T}\alpha^{s-1}(L_{i,s}-C_{i,s}) \qquad (2)$$

PV_2 is present value of net cash flows from <u>new</u> products, without interactions

$$= \sum_{i=m+1}^{m+n}\sum_{t=1}^{T}X_{i-m,t}\sum_{s=t}^{T}\alpha^{s-1}[L_{i,s-t+1}-C_{i,s-t+1}] \qquad (3)$$

PV_3 is present value of sales revenue due to interactions among pairs of existing products

$$= \sum_{i=2}^{M}\sum_{j=1}^{i-1}\sum_{t=1}^{T}\alpha^{t-1}(d_{j,i,t}\cdot L_{j,t}+d_{i,j,t}\cdot L_{i,t})Z_{i,j,t} \qquad (4)$$

PV_4 is present value of interactions effects among existing and new products

$$= \sum_{i=m+1}^{m+n}\sum_{j=1}^{m}\sum_{t=1}^{T}\alpha^{t-1}d_{j,i,t}L_{j,t}Z_{i,j,t} + \sum_{i=m+1}^{m+n}\sum_{j=1}^{m}\sum_{t=1}^{T}\sum_{u=1}^{t}\alpha^{t-1}d_{i,j,t-u+1}L_{i,t-u+1}W_{i,j,t,u} \qquad (5)$$

PV_5 is present value of interaction effects among new products

$$= \sum_{i=m+2}^{m+n}\sum_{j=m+1}^{i-1}\sum_{t=1}^{T}\sum_{u=1}^{t}\alpha^{t-1}d_{j,i,t-u+1}L_{j,t-u+1}W_{j,i,t,u} + \sum_{i=m+2}^{m+n}\sum_{j=m+1}^{i-1}\sum_{t=1}^{T}\sum_{u=1}^{t}\alpha^{t-1}d_{i,j,t-u+1}L_{i,t-u+1}W_{i,j,t,u} \qquad (6)$$

Constraints

Each new product can be introduced only once:

$$\sum_{t=1}^{T}X_{j,t}\le 1, \quad j=1,\dots,n. \qquad (7)$$

Each existing product can be withdrawn only once:

$$\sum_{t=1}^{T}Y_{j,t}\le 1, \quad j=1,\dots,m. \qquad (8)$$

The following constraints must be imposed to create $Z_{i,j,t}$:

For interaction terms, within existing product mix,

$$\sum_{k=1}^{t}Y_{i,k}+\sum_{k=1}^{t}Y_{j,k}+Z_{i,j,t}\ge 1$$
$$\sum_{k=1}^{t}Y_{i,k}+\sum_{i=1}^{t}Y_{j,k}+2Z_{i,j,t}\le 2 \qquad (9)$$

for $2\le i\le m$, $j<i$; $1\le t\le T$.

For interaction terms between existing and new products,

$$\sum_{k=1}^{t}X_{i-m,k}-\sum_{k=1}^{t}Y_{j,k}-Z_{i,j,t}\le 0$$
$$\sum_{k=1}^{t}X_{i-m,k}-\sum_{k=1}^{t}Y_{j,k}-2Z_{i,j,t}\ge -1 \qquad (10)$$

for $m<i\le m+n$; $1\le j\le m$; $1\le t\le T$.

For interaction terms within new products,

$$\sum_{k=1}^{t}X_{i-m,k}+\sum_{k=1}^{t}X_{j-m,k}-Z_{i,j,t}\le 1$$
$$\sum_{k=1}^{t}X_{i-m,k}+\sum_{k=1}^{t}X_{j-m,k}-2Z_{i,j,t}\ge 0 \qquad (11)$$

for $i=m+1,\dots,m+n, m<j<i, t=1,\dots,T$.

The following constraints must be imposed to crease $W_{i,j,t,u}$:

$$Z_{i,j,t}+X_{i-m,u}-W_{i,j,t,u}\le 1 \text{ if } i>j$$
$$Z_{i,j,t}+X_{i-m,u}-2W_{i,j,t,u}\ge 0$$
and
$$Z_{j,i,t}+X_{i-m,u}-W_{i,j,t,u}\le 1 \text{ if } i<j \qquad (12)$$
$$Z_{j,i,t}+X_{i-m,u}-2W_{i,j,t,u}\ge 0$$

for $i=m+1,\dots,m+n$; $j=1,\dots,m+n$; $i\ne j$; $t=1,\dots,T$; $u=1,\dots,t$.

Other constraints may be introduced to reflect budget, profitability, technology, production capacity, and manpower restrictions as well as product dependencies. For example, introduction or withdrawal of one product can be made contingent upon introduction or withdrawal of any other product or group of products. If management wants to restrict the <u>p-th</u> existing product from being on the market at the same time as the <u>g-th</u> new product, perhaps due to production capacity limitations, the constraints

$$\sum_{t=1}^{s}Y_{p,t}-\sum_{t=1}^{s}X_{q,t}\ge 0 \text{ for } s=1,\dots,T$$

will ensure this.

DISCUSSION OF THE MODEL

The model does not permit reintroduction of existing products once they have been withdrawn, or withdrawal within the planning horizon of the new products once introduced.

It is assumed that all withdrawals and introductions are made at the beginning of a period. The number of periods in the planning horizon, T, and the length of the periods in the model will depend on the type of products for which the model is being formulated and on the extent of planning the company wishes to do. The length should be at least as long as the typical life cycle of the products under consideration and preferably twice as long so that the potential of the products introduced towards the end of the first half of the horizon is fully realized within the horizon period. An increased number of periods in the horizon means an increasing number of variables in the problem formulation and higher computational costs. In addition, errors induced by the increased uncertainty in parameter values for periods far into the future may more than offset the advantages of a long planning horizon.

The model requires cost, revenue, and interaction estimates before it can be used. $L_{i,t}$ defines the estimated revenue from the <u>i</u>th product in the <u>t</u>th period after its introduction, independent of all interactions. All existing products are assumed to have been introduced in the first period. Similarly, $C_{i,t}$ defines the corresponding total costs. The difference, $(L_{i,t}-C_{i,t})$, represents the estimated net cash flow if product <u>i</u> is in the product mix in the <u>t</u>th period after its introduction.

The cost, $C_{i,t}$, is assumed to be free of interaction effects. Under certain circumstances, such as when production is subcontracted, this is a reasonable assumption for production costs. However, cost interactions can be modeled by adding cost interaction terms similar to the revenue interactions in the objective function segments.

When a direct costing approach is used the firm can trace fixed and variable costs to the various products. Such an approach would involve contribution margins in the objective function, and would also permit the inclusion of interactive marketing costs. The objective would be to maximize the contribution to the covering of common costs and profits [2].

The revenue estimates require price and volume forecasts. An appropriate estimation technique is presented in [1], and is adapted to a discounted cash flow approach in [3]. These techniques require subjective estimates from management.

When a product is withdrawn in period t, no further costs or revenues are incurred. Product life cycles longer than $\underline{T}$ are effectively truncated at period $\underline{T}$. This is not a serious drawback when it is considered that period $\underline{T}$ lies as far in the future as the company would like to plan.

The model assumes that management is able to determine the various marketing mix variables for each product individually. If this has not been done, a product can be represented in the model by a set of products, each corresponding to a different marketing mix. These multiple products may be constrained to be mutually exclusive so that only one may enter the product mix. In this way the model can also determine marketing mix decisions.

The interaction estimates between each pair of products could be obtained by questioning the managers concerned with those products. Though these estimates are essentially subjective, their use in this model should yield better decisions than intuitive determination of product introduction and withdrawal.

The matrix of interactions $\underline{d}$ has dimension ((m+n)x(m+n-1) xT). These interactions can be modeled in a linear framework by the addition of the interaction variables Z and W defined above. Each Z and W variable requires two constraints in its definition.

The total number of variables in this model is at most

$$\frac{1}{2} T(m+n)(m+n+1) = \frac{T(T+1)}{2} (m+n-1)n$$

and the total number of necessary constraints is at most $(m+n)\ \{1+T(m+n)\} + T(T+1)n \cdot (m+n-1)$.

In addition to products already on the market and products available for immediate introduction, products in the R&D stage and competitor's products can be included in the model. At each stage of R&D costs, revenues, and product interactions can be estimated for proposed products. These products can be constrained not to be introduced into the market before the expected development period is over. Results of the model can be used to determine whether R&D work on the product should be continued, slowed down, speeded up or dropped altogether.

Similarly, the competition's products can be included in the model and constrained to stay on the market as long as expected. Both the revenues and the costs of such products will be zero. The interaction terms depict the effect the competitor's products have on the firm's product line. Those products which the competition is expected to introduce in the future can be included in the model by constraining them to enter in the appropriate future period. Optimal product mix decisions obtained from a model which includes the competitive products would be superior to the decisions made without considering their effects.

It is suggested that this model be run at monthly to semi-annual intervals depending on the size of the problem, computation costs, and volatility of the conditions in the R&D lab and the market. Each run will indicate an introduction and withdrawal plan for the periods over the horizon and a GO, CONTINUE, NOGO signal for the R&D products included in the model. Even new project proposals for R&D might be included in the model for evaluation. As revised plans become available, appropriate marketing decisions can be made to implement them over the horizon period. Thus a periodic application of this model will provide a long term as well as short term planning tool.

Subjective or non-quantifiable factors can be considered by using a procedure developed by Piper and Zoltners [6] which makes it possible to obtain the set of best solutions to an integer programming problem. The decision maker can then make his choice among these solutions in a subjective way considering all factors excluded from the formal model.

A SPECIFIC PROBLEM

Progressive Industries, Inc., (PII), is a medium sized firm which manufactures and sells electric blenders. Ideas for developing new models and products come from its own research and development facility. Beginning with the initial idea for a product, PII makes and reviews a series of decisions: whether to begin development, to continue development, to market, and finally, to withdraw the product.

At present PII markets two different blender models, a half-gallon family size (model A) introduced three years ago and a one quart size (model B) introduced one year ago. The marketing manager estimates the remaining market life of model A to be three years and of model B to be four years. The marketing manager also believes that the two blenders are functionally substitutes for each other. Although the blenders are sold to different market segments (segmented by family size), sales revenues for each product when they are marketed jointly are estimated to be about 10% less than the estimated sales of each if they are marketed independently. The estimated sales revenues for the next five years are given in the first two rows of Table 1. Based on past performance, PII has determined its production and marketing costs for these products are as given in the first two rows of Table 2.

TABLE 1

Expected Revenues Over the Product Life of Each Model ($10,000)

Model	Year 1	2	3	4	5
A	10	13	16	0	0
B	20	15	10	5	0
New Deluxe Blender	5	8	18	14	10
New Mixer	3	12	25	18	6

TABLE 2

Expected Costs Over the Product Life of Each Model ($10,000)

Model	Year 1	2	3	4	5
A	7.0	8.5	10.0	1.0	1.0
B	13.0	10.5	8.0	4.5	1.0
New Deluxe Blender	8.5	7.0	12.0	10.0	8.0
New Mixer	11.5	8.0	14.5	11.0	5.0

R&D recently completed designs for two new products--a deluxe one quart blender to appeal to a high income market, and a new addition to the product line, a mixer.

Initial market research has indicated that each of these products could be marketed profitably by itself. The estimated revenues for these two new products, if each were to be marketed alone, are given in the last two rows of Table 1, with the corresponding total costs in Table 2. The marketing manager believes that introducing the deluxe blender would not affect the sales of model A, but the presence of model A on the market would decrease the expected revenues of the blender by 10%. If the deluxe blender is introduced, the sales of model B are expected to fall approximately 25% while the presence of model B on the market is expected to decrease the expected revenues of the blender by 20%.

The mixer, however, is expected to increase the sales of models A and B by about 10% because a greater variety of products would make the brand name more appealing to the market. The presence of model A would result in a corresponding increase in appeal for the brand name and would lead to an estimated 10% increase in the mixer revenues. It was thought that model B would have similar effect. If both the deluxe blender and mixer are present on the market simultaneously, sales of the mixer would be 5% higher than expected. The presence of the mixer on the market is not expected to affect the sales of the deluxe blender. Table 3 summarizes the product revenue interactions.

The problem faced by management is to schedule the introduction of the deluxe blender and the mixer, and subsequent withdrawal of existing products so as to maximize the net present value of the product line over the planning horizon.

The model developed in the previous section was applied to this problem resulting in an integer program with 140 variables and 244 constraints. Fortunately, integer programs of this type possess a structure which considerably reduces solution effort. For example, the majority of constraints are logical constraints, like (7) - (12), which eliminate many solutions and can be handled explicitly within an algorithm. In a typical problem there will be comparatively few structural constraints such as budgeting, production capacity. Further advantage arises because the objective function "cumulative" profit coefficients are monotonically nondecreasing as a function of $\underline{t}$. Consequently, if the revenue interaction terms were rearranged, they, too, would be monotonically changing as a function of $\underline{t}$.

Although the PII product mix problem could be solved by hand, we used an all purpose integer programming code developed by Piper [2] which easily solved the problem. The optimal solution, shown in Table 4, indicated that PII can make a profit of $521,000 over the next five years if

1. Model A is withdrawn from the market at the beginning of period 5,
2. Model B is withdrawn from the market at the beginning of period 5,
3. the new deluxe blender is never introduced to the market, and
4. the new mixer is introduced to the market at the beginning of period 1.

TABLE 3

Product Revenue Interactions for the Product Line

Model	Model: A	B	Deluxe Blender	Mixer
A	0	-0.10	0	0.10
B	-0.10	0	-0.25	.10
Deluxe Blender	-0.10	0.20	0	0
Mixer	.10	.10	.05	0

TABLE 4

Product Revenues and Optimal Product Mix Profits
($10,000)

Model	Year: 1	2	3	4	5	Total
A	10	13	16	0	*	39
B	20	15	10	5	*	50
Blender	*	*	*	*	*	-
Mixer	3.6	14.4	30	21.6	6	75.6
Total	33.6	42.4	56.0	26.6	6	164.6
Total Costs#	31.5	27.0	32.5	16.5	5	112.5
Profits	2.1	12.4	23.5	10.1	1	52.1

*Not in the product mix
#From Table 2

It is interesting to observe that model A is kept on the market in the fourth year when its sales are zero. This can be explained by noticing that model A has a positive revenue interaction with the new mixer. Mixer sales, without revenue interactions in the fourth period are estimated to be $180,000 (Table 1). Total sales will increase by $36,000 if model A stays on the market. Since model A accounts for $18,000 of this increase which is larger than model A's production costs and other interaction effects, model A will remain on the market in period four. If management is uneasy about this solution, model A can be withdrawn at the beginning of period four, producing cost savings of $10,000, sacrificing revenues of $18,000 for the mixer, and increasing model B's revenues by $5,000. Thus, dropping model A would reduce profits by $3,000 to $518,000.

After inspecting the suggested solution, some questions may arise concerning the new deluxe blender. For example: Why wasn't the deluxe blender introduced? How would its introduction impact on expected profits? What is the best introduction and withdrawal strategy if management required the introduction of the new deluxe blender? These questions can be resolved via postoptimality analysis.

The new deluxe blender was not introduced because of its negative revenue interaction with models A and B. This can be verified when one observes that if interaction effects were not considered the optimal solution would differ from the previous optimum in that model A would be withdrawn one period sooner and the new deluxe blender would be introduced in the first period. The estimated profit under this plan is $399,000. See Table 5.

TABLE 5
Product Revenues and Product Mix Profits
Without Revenue Interactions
($10,000)

Model	Revenues — Year: 1	2	3	4	5	Total
A	10	13	16	*	*	39.0
B	15	11.25	7.5	4.25	*	38.0
Blender	3.5	5.6	12.6	11.2	10	42.9
Mixer	3.75	15.0	31.25	20.7	6.3	77.0
Total	32.25	44.85	65.35	36.15	16.3	196.9
Total Costs#	40.00	34.00	44.50	25.50	13.0	157.0
Profits	(7.75)	10.85	22.85	10.65	3.3	39.9

*Not in the product mix
#From Table 2

If management felt that the blender should be introduced, the optimal product-mix strategy is shown in Table 6. Estimated profit over the five periods is $453,000. This is a decrease of $68,000 from the previous optimum. Hence, a new deluxe blender introduction would cost PII $68,000 in profits.

Table 7 shows the value of the new mixer to the firm's product mix. If the new mixer was withheld from the market, expected profits would be reduced to $237,000.

Although the product mix example used was simple, nevertheless, as was shown, the model is easily adaptable to explore the implications of alternative decisions. As illustrated, a number of "what if" questions can be answered, thereby allowing management to understand the consequences of alternative solutions.

CONCLUSION

This paper has considered the problem of determining a firm's optimal product mix. A 0-1 integer programming model has been presented which considered revenue interactions between products. Products already on the market

TABLE 6

Product Revenues and Product Mix Profits With Blender ($10,000)

Model	Revenues: Year 1	2	3	4	5	Total
A	10	14.3	17.6	*	*	41.9
B	15	*	*	*	*	15.0
Blender	3.5	7.2	16.2	14	10	50.9
Mixer	3.75	13.8	28.75	18.9	6.3	71.5
Total	32.25	35.3	62.55	32.9	16.3	179.3
Total Costs#	40.00	23.5	36.50	21.0	13.0	134.0
Profits	(7.75)	11.8	26.05	11.9	3.3	45.3

*Not in the product mix
#From Table 2

TABLE 7

Product Revenues and Product Mix Profits Without Mixer ($10,000)

Model	Revenues: Year 1	2	3	4	4	Total
A	9	13	16	*	*	38
B	18	*	*	*	*	18
Blender	*	4.5	7.2	18	14	43.7
Mixer	*	*	*	*	*	-
Total	27	17.5	23.2	18	14	99.7
Total Costs#	20	15.5	22.0	10	8	75.5
Profits	7.0	2.0	1.2	8	6	24.2

*Not in the product'mix
#From Table 2

may be withdrawn and new products may be added, over a multi-period horizon.

The model considers only first order (between each pair of products) interactions. Higher order interactions can be incorporated at the cost of substantial increase in problem size and data requirements.

Since there is no concept of duality in integer programming, sensitivity of solution to variation in input parameters has yet to be determined. If solutions given by the technique can be shown to be robust with respect to small changes in estimates of parameters, accuracy requirements on data collection can be accordingly relaxed.

Finally, it should be stressed that, while complicated, the integer programming model presented in this paper conceptualizes the basic combinatorial nature of the multi-period product mix problem. Actual operationalization would involve specialized computer codes which would take considerable advantage of the problem structure. These codes would make it feasible to develop interactive decision systems for the marketing executive.

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