Chapter 102
Positive Interest Rates and Yields: Additional Serious Considerations*

Jonathan Ingersoll

Abstract Over the past quarter century, mathematical modeling of the behavior of the interest rate and the resulting yield curve has been a topic of considerable interest. In the continuous-time modeling of stock prices, one only need specify the diffusion term, because the assumption of risk-neutrality for pricing identifies the expected change. But this is not true for yield curve modeling. This paper explores what types of diffusion and drift terms forbid negative yields, but nevertheless allow any yield to be arbitrarily close to zero. We show that several models have these characteristics; however, they may also have other odd properties. In particular, the square root model of Cox–Ingersoll–Ross has such a solution, but only in a singular case. In other cases, bubbles will occur in bond prices leading to unusually behaved solutions. Other models, such as the CIR three-halves power model, are free of such oddities.

Keywords Term structure • Interest rates • Price bubbles • Positivity

102.1 Introduction

In a recent paper Pan and Wu (2006) have posited that (nominal) interest rates of all maturities should have a lower bound of exactly zero – that is, yields arbitrarily close to zero should be possible for bonds of all maturities. They derived the "unique" model in which this assumption together with continuity and the absence of arbitrage are satisfied. This paper questions that desirata and highlights some curiosities that models possessing this property also have. In addition, it shows that, far from there being a unique model with this property, there are in fact countless other models that satisfy these conditions even if only one source of risk is assumed.

Section 102.2 of this paper briefly discusses the relation among yields and highlights the question of a zero lower bound. Section 102.3 reviews the Cox et al. (1985) and Pan Wu (2006) models and shows that the zero lower bound in the latter is due to the interest rate process, which has an absorbing barrier at zero and no finite-variance steady-state distribution. Section 102.4 derives the bubble-free solution to the Pan Wu model when the risk-neutral and true processes differ in their behavior at zero. Section 102.5 develops a two-state variable extension to the Cox, Ingersoll, Ross model which permits a lower bound of zero for all yields with no price bubbles and with no absorption of the interest rate at zero. Section 102.6 discusses non-affine models of yields and shows it is possible to have interest rate models with a finite-variance steady-state distribution, no absorption at zero, and a lower bound of zero for all yields. Section 102.7 briefly discusses other constant lower bounds for yields.

102.2 A Non-Zero Bound for Interest Rates

Pan and Wu argue: "Asserting that an interest rate can be negative or cannot be lower than, say, 3%, is equally absurd. For example, no rational traders are willing to offer free floors at any strictly positive level of interest rates." We contend that these claims are vastly different. The presence of cash alone requires that nominal interest rates of all maturities be nonnegative. On the other hand, the lack of interest rate floors is just common sense. It is true that the Cox–Ingersoll–Ross interest rate model and other similar models absolutely prohibit yields to various maturities below certain non-zero levels so that interest rate floors at some positive interest rate should be cost-free. But why would any trader offer a zero-cost floor at a rate he knew the interest rate could never reach? True, he could not lose on such a contract, but there would be no possibility of gain either if they were being given for free. Conversely, no buyer would be willing to pay any positive

J. Ingersoll (✉)
Yale School of Management, New Haven, CT, USA
e-mail: jonathan.ingersoll@yale.edu

* The author has benefited from his discussions with his colleagues. This paper is reprinted from *Advances in Quantitative Analysis of Finance and Accounting*, 7 (2009), pp. 219–252.

price and would completely indifferent about receiving such a floor at a zero cost. There would simply be no market for such contracts.

In any case we must not forget that interest rate models are models, that is, simplifications of the world. By their very nature, all models make absolute claims of one type or another. The model that Pan and Wu derived, for example, requires that the 5-year yield to maturity always be greater than (or less than depending on the model's two parameters) the 3-month yield to maturity. But no trader would quote swap rates based on this guarantee.

If a model that is based on reasonable assumptions makes a surprising prediction that should be taken as a good sign. We want our models to tell us things we didn't know or otherwise lead to new intuitions. The model's assumptions may turn out to be wrong, but the logical process that leads us to the surprising conclusion is valuable nonetheless. So the appropriate question becomes: Is the conclusion that yields to maturity possess strictly positive bounds a surprising one? I do not think so and believe just the opposite in fact.

Historically, yield curves have been downward sloping when the spot rate is high and upward sloping when the spot rate is low. This is generally explained by saying that long rates are related to expected spot rates and that the spot rate has some long-run or steady state distribution so that the expected rate is always closer to the long-run average than is the current spot rate. This notion can be made precise as follows.

Continuous-time interest rate models generally assume that bonds can be priced using a risk-neutral (or equivalent martingale) process[1] and a form of the expectations hypothesis for discounting. In particular the price of a zero-coupon bond is given by

$$P_t(r,\tau) = \widehat{\mathbb{E}}_t\left[\exp\left(-\int_t^{t+\tau} r_s ds\right)\right]. \qquad (102.1)$$

Here $\widehat{\mathbb{E}}_t$ denotes the expectation at time t with respect to the equivalent martingale process, and r_s is the instantaneous spot rate prevailing at time s. When bonds are priced like this, the yield to maturity is the negative geometric expectation of the instantaneous rates over the time interval

$$\begin{aligned} Y_t(r,\tau) &\equiv -\frac{1}{\tau}\ell\text{n}\, P_t(r,\tau) \\ &= -\frac{1}{\tau}\ell\text{n}\left\{\widehat{\mathbb{E}}_t\left[\exp\left(-\int_t^{t+\tau} r_s ds\right)\right]\right\}. \qquad (102.2) \end{aligned}$$

[1] When pricing bonds and other fixed-income assets in the presence of interest rate uncertainty, the equivalent martingale process that allows discounting at the interest rate does not result from assuming risk-neutrality on the part of investors as it does in the Black–Scholes model. Nevertheless, the term "risk-neutral" is still commonly applied. See Cox et al. (1981) for further discussion of this matter.

If the instantaneous rate is bounded below by zero but free to move above zero, the usual properties of averages would seem to guarantee that long-term yields would be bounded away from zero with the exact bound depending on the properties of the stochastic process generating the evolution of the interest rate and the maturity of the bond in question. Apparently it would not be surprising to find models with yields bounded away from zero; quite the contrary, we should expect just that property.

In fact were the average in Equation (102.2) an arithmetic one, this property would be universally true and long rates would be bounded away from zero whenever positive interest rates remained *possible* in the future. Geometric averages are different, though. A negative geometric average is never greater than the corresponding arithmetic average, and if there is any variation in the random variable it is strictly less.[2] So yields may have lower bounds of exactly zero even when the instantaneous rate is guaranteed to be remain positive.

Before going on to explore positive interest rates in more detail we review the Cox et al. (1985) and the special case of it, which is the Pan and Wu term (2006) structure models in the next section.

102.3 The Cox–Ingersoll–Ross and Pan–Wu Term Structure Models

Though derived in a different fashion, the Pan–Wu (henceforth PW) model is a special case of the Cox–Ingersoll–Ross (henceforth CIR) affine term structure mode in which the risk-neutral dynamics of the instantaneous interest rate are

$$dr_t \overset{\wedge}{=} \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}\,d\omega_t. \qquad (102.3)$$

The symbol $\overset{\wedge}{=}$ is used as a reminder that the dynamics in Equation (102.3) are the equivalent-martingale "risk-neutral" dynamics, the pricing equation for all zero-coupon bonds is

$$0 = \tfrac{1}{2}\sigma^2 r P_{rr} + \kappa(\theta - r)P_r - rP - P_\tau \qquad (102.4)$$

where $\tau \equiv T - t$ is the time until the maturity date T. The CIR solution is

$$P(r,\tau) = A(\tau)\exp[-B(\tau)r] \qquad (102.5)$$

[2] In discrete time, the yield in Equation (102.2) is $1 + Y \equiv \left(\hat{\mathbb{E}}[((1+r_1)\cdots(1+r_n))^{-1}]\right)^{-1/n}$. By Jensen's inequality this is less than $\hat{\mathbb{E}}[((1+r_1)\cdots(1+r_n))^{1/n}]$. So one plus the yield to maturity is less than the expectation of the geometric average of one plus the future prevailing spot rates. Since a geometric average is never larger than the corresponding arithmetic average, the yield to maturity must be less than the risk-neutral expected spot rate prevailing in the future.

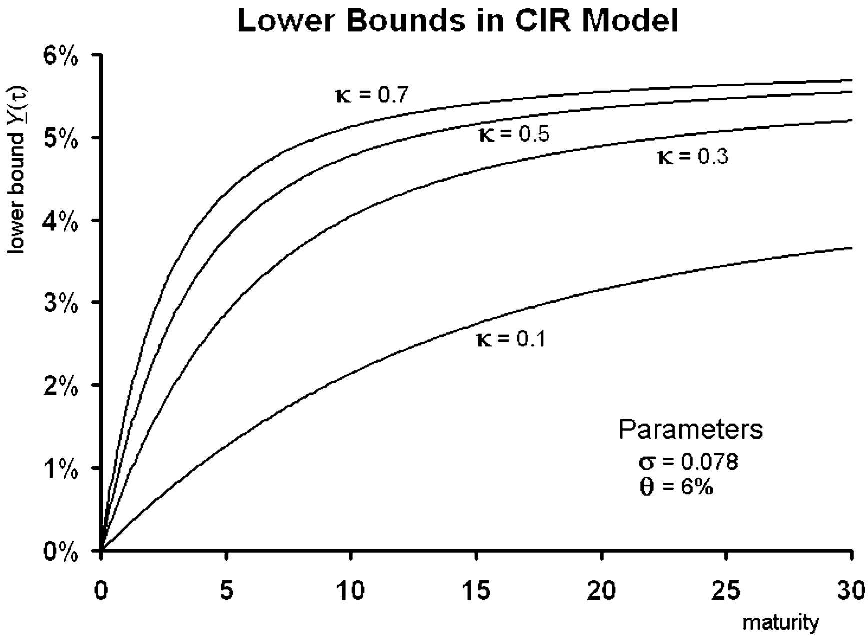

Fig. 102.1 Lower bound for CIR yield curves. In the CIR model, the yield to maturity for any bond is $Y(r,\tau) = -\ell n[A(\tau)]/\tau + B(\tau)r/\tau$. The τ period yield is increasing in the spot rate, r, and has a lower bound of $\underline{Y}(\tau) = -\ell n[A(\tau)]/\tau$, which is achieved when the spot rate is zero. For the PW model, $A(\tau) \equiv 1$ and all yields have a lower bound of zero. The parameter $\sigma = 0.078$ gives a standard deviation of changes in the interest rate equal to 1.9% point at the mean interest rate level of $\theta = 6\%$. The parameter κ measures the rate of return towards the mean level. After t years the interest rate will on average have moved the fraction $e^{-\kappa t}$ of the distance back toward θ

$$\text{where} \quad B(\tau) \equiv \frac{2(1-e^{-\gamma\tau})}{2\gamma + (\kappa-\gamma)(1-e^{-\gamma\tau})}$$

$$A(\tau) \equiv \left[\frac{2\gamma e^{(\kappa-\gamma)\tau/2}}{2\gamma + (\kappa-\gamma)(1-e^{-\gamma\tau})}\right]^{2\kappa\theta/\sigma^2}$$

$$\gamma \equiv \sqrt{\kappa^2 + 2\sigma^2}.$$

The PW derivation is substantially different from that used by CIR. They start with a general vector diffusion, $\mathbf{u}_t$, and then to ensure positive interest rates assume that yields of all maturities have a quadratic form $y(\mathbf{u}_t,\tau) = \frac{1}{\tau}\mathbf{u}_t'\mathbf{W}(\tau)\mathbf{u}_t$ where $\mathbf{W}$ is a symmetric positive definite matrix. They go on to show that the only model satisfying these conditions is equivalent to the one-factor, two-parameter model given in Equation (102.3) and (102.5) with $\theta = 0$. In particular, for the PW model, $A(\tau) \equiv 1$.

In the CIR and PW models, the yield to maturity for a zero-coupon bond of maturity τ is

$$Y(r,\tau) \equiv -\frac{1}{\tau}\ell n\, P(r,\tau) = -\frac{\ell n\, A(\tau)}{\tau} + \frac{B(\tau)}{\tau}r. \qquad (102.6)$$

Since $A(\tau) \leq 1,\ B(\tau) \geq 0$, and the interest rate cannot go below zero for the assumed dynamics, the τ-maturity yield is clearly bounded below by $\underline{Y}(\tau) \equiv -\ell n(A(\tau))/\tau$, which is strictly positive except when $\theta = 0$, the PW case.

The limiting yield to maturity for an infinitely lived zero-coupon bond is $2\kappa\theta/(\kappa+\gamma)$, which is somewhat less than θ. Since this quantity is independent of the current interest rate, it is also the lower bound for the infinite-maturity zero-coupon rate, which is constant in the CIR model. As shown in Fig. 102.1, the lower bounds for shorter maturity rates increase from zero (at $\tau = 0$) to approach this level asymptotically. The rate of increase in the lower bounds is governed primarily by the parameter κ with larger values producing a faster approach and generating higher lower bounds for yields of all maturities.[3]

The difference between the lower bounds in the two models is due to the behavior of the interest rate at zero. For the CIR model with $\theta > 0$, zero is a natural reflecting barrier of the interest rate process. If the interest rate hits zero, the uncertainty momentarily vanishes, and the then certain change in the interest rate is an increase – immediately moving it back to a positive value. For the PW model, $r = 0$ is an absorbing barrier. The uncertainty still vanishes, but now the expected future change is also zero, so the interest rate remains stuck at that level. Once the instantaneous interest rate reaches zero, it and the yields of all maturities are zero

[3] The lower bounds for all yields also increase with κ because the asymptotic value, $Y_\infty = 2\kappa\theta/(\kappa+\gamma)$, does. See Dybvig, et al. (1996) for an analysis of the asymptotic long rate.

forever. It is obvious, therefore, that no yield can have a positive lower bound. Furthermore, this property is true in any model of interest rates in which zero is an accessible absorbing barrier for the interest rate.[4]

A closely related feature of the PW model is that the values of zero-coupon bonds do not approach zero for long maturities. $B(\tau)$ is an increasing function but is bounded above by $2/(\kappa + \gamma)$, so no zero-coupon bond has a price less than $\exp[-2r/(\kappa + \gamma)]$ regardless of its maturity. In other words, no zero-coupon bond has a value less than the value that a $2/(\kappa + \gamma)$ year bond would have were the yield curve flat at r. For the parameters estimated by PW this "maximal" maturity is about 50 years. This limit also means that the prices for annuities and coupon bonds with fixed coupons will become unbounded as the maturity increases. This aspect of the model is certainly problematic in many other contexts as well since transversality will be violated for many valuation problems.

Since the interest rate can be trapped at zero, it is not surprising that long bond prices do not vanish. However, this problem does not follow directly from the interest rate being trapped at zero. It can be true in other models as well. For example, in Dothan's (1978) model of interest rates, long-term bond prices also do not approach zero, although the interest rate has a lognormal distribution for which zero is inaccessible.[5] It is also true in the two-factor version of the CIR model discussed in Sect. 102.4 below.

102.4 Bubble-Free Prices

An interesting question remains. Can yields have a lower bound of zero, even when the interest rate cannot be trapped at zero and the values of zero-coupon bonds do become vanishingly small for long maturities? At first the answer to this question would seem obviously to be yes. Pricing is based on the risk-neutral process, but whether or not the interest rate is trapped at zero is a property of the true process. Apparently all that would be required is that the true process not have an absorbing state at zero while the risk-neutral process did. For example, the true and risk-neutral processes could be

$$\begin{aligned} dr_t &= \mu(r)dt + \sigma\sqrt{r_t}d\omega_t \qquad \mu(0) > 0 \\ dr_t &\overset{\wedge}{=} -\kappa r_t dt + \sigma\sqrt{r_t}d\omega_t. \end{aligned} \tag{102.7}$$

With $\mu(0) > 0$, zero will be a natural reflecting barrier for the interest rate process just as it is in the CIR model. Whenever the interest rate reaches zero, it will immediately become positive again. As always, the diffusion term in the risk-neutral process is identical to that in the true process, but the drift term is altered.

Unfortunately, the true and risk-neutral processes in Equation (102.7) are not equivalent as is required for a proper risk-neutral process. The true process results in a continuous probability density for r defined over all nonnegative values.[6] The risk-neutral process also has a continuous distribution over all positive r – in particular a non-central chi squared distribution, but there is an atom of probability at zero as well. As shown in the appendix, the risk-neutral probability that the interest rate will have reached zero and been trapped there at or before time T is

$$\widehat{\Pr}\{r_T = 0 | r_t = r\} = \exp\left(-2\kappa r/[\sigma^2(e^{\kappa(T-t)} - 1)]\right). \tag{102.8}$$

There is a corresponding positive state price atom (not simply a positive state price density) associated with the state $r_T = 0$. The Arrow–Debreu price for the state (atom) $r_T = 0$ when $r_t = r$ is[7]

$$Q(r,t) = \exp\left(\{\kappa - \gamma\coth[\tfrac{1}{2}\gamma(T-t)]\}\, r/\sigma^2\right). \tag{102.9}$$

But under the true process, $r_T > 0$ with probability one, so the true state price (like the true probability) cannot have a positive atom for $r_T = 0$.

As previously stated the problem here is a lack of equivalence between the true and risk-neutral measures. They do not possess the same probability-zero sets of states. There are, in fact, two opposite situations to consider. If zero is inaccessible under the true process, then the risk-neutral process clearly has a larger set of possibilities. On the other hand, if zero is accessible under both processes, then the

[4] See, for example, Longstaff (1992), which solves the bond-pricing problem in the CIR square-root framework with $r = 0$, an absorbing barrier.

[5] In Dothan's 1978 model, interest rates evolves as $dr = \sigma r d\omega$. The asymptotic bond price is $\sqrt{8r}K_1\left(\sqrt{8r}/\sigma\right)/\sigma > 0$, where K_1 is the modified Bessel function of the second kind of order one. See, in particular, Fig. 102.2 on p. 66.

[6] The drift term, $\mu(\cdot)$, must satisfy mild regularity conditions. If μ is continuous and the stochastic process is not explosive so that $r = \infty$ is inaccessible, then the density function for $r \in (0, \infty)$ will exist for all future t with a limiting steady state distribution of

$$\frac{c}{\sigma^2 r}\exp\left[2\sigma^{-2}\int^r \mu(x)x^{-1}dx\right]$$

where c is chosen to ensure it integrates to unity. The density function may of course be zero for some values.

[7] The function $\coth(x)$ is the hyperbolic cotangent: $\coth x \equiv (e^x + e^{-x})/(e^x - e^{-x})$. The hyperbolic functions are related to the standard circular functions as:

$$\sinh x = -i\sin ix, \cosh x = \cos ix, \tanh x = -i\tan ix,$$
$$\text{and } \coth x = -i\cot ix.$$

true process has a large set of possibilities – namely those in which the interest rate reaches the origin and becomes positive again.

The latter case, when zero is accessible but not an absorbing barrier under the true process, is irreconcilable with the assumed risk-neutral process. The risk-neutral process completely specifies the partial differential equation and this cannot be altered to modify the probabilities for "interior" states after the origin has been reached and left. In the former case, when zero is inaccessible, the risk-neutral process can be reconciled with the true process making the measures equivalent by assigning zero probability to the offending $r_T = 0$ state. This can be done because the offending state is at a boundary of the distribution so the assignment can be handled by an appropriate boundary condition to the partial differential Equation (102.4) leaving it and therefore the risk-neutral diffusion process itself unchanged.

In the remainder of this section we assume that zero is inaccessible under the true process. A sufficient condition for zero to be inaccessible is that $\mu(r) \geq \frac{1}{2}\sigma^2$ for all r sufficiently small.[8]

The boundary condition applied at $r = 0$ to the bond pricing problem clearly has the form $P(0,\tau) = p(\tau)$ for some function $p(\tau)$. In the PW model, $p(\tau) = 1$ is used, but other assumptions can be made without changing the local no-arbitrage condition inherent in the partial differential equation. Furthermore, since zero is inaccessible, the condition will never actually apply. But what function is logically consistent? Fix a particular boundary condition, $p(\tau)$, by conjecture. Now consider a contract that grants $1/p(\tau)$ zero-coupon bonds when the interest rate is zero and then terminates. Such a contract is clearly worthless, if $r = 0$ is truly inaccessible. However, the risk-neutral pricing procedure will assign a price equal to the expected present value of receiving $[1/p(\tau)] \cdot p(\tau) = 1$ when r reaches zero. This is just the value of Q as given in Equation (102.9) above. Therefore, we must have $p(\tau) = 0$. We shall refer to this boundary condition and the resulting solution as bubble-free for reasons to be explained below.

As shown in Appendix 102A, the value of a zero-coupon bond under the true and risk-neutral processes in Equation (102.7) and the correct no-arbitrage boundary condition $P(0,\tau) = 0$ when zero is inaccessible is

$$P^{\mathrm{BF}}(r,\tau) = \exp[-rB(\tau)]\,(1 - \exp[-r\xi(\tau)])$$
$$\text{where } \xi(\tau) \equiv \frac{2\gamma^2}{\sigma^2}[\gamma\sinh(\gamma\tau) + \kappa\cosh(\gamma\tau) - \kappa]^{-1}. \tag{102.10}$$

The function $B(\tau)$ and the parameter γ are the same as in the CIR and PW models so this formula is just the PW price multiplied by the factor $1 - \exp[-\xi(\tau)r]$. Since $\xi(\tau) > 0$ for $\tau > 0$, the value in Equation (102.10) is strictly less than the PW model price at any time before the bond matures.

The yield to maturity computed from Equation (102.10) is

$$\begin{aligned} Y^{\mathrm{BF}}(r,\tau) &\equiv -\frac{1}{\tau}\ell\mathrm{n}\, P^{\mathrm{BF}}(r,\tau) \\ &= \frac{B(\tau)}{\tau}r - \frac{1}{\tau}\ell\mathrm{n}\,(1 - \exp[-r\xi(\tau)])\,. \end{aligned} \tag{102.11}$$

Since bond prices are lower than in the PW model, yields are correspondingly higher. However, as shown in Fig. 102.2, the yield curves are very similar for short maturities particularly when the spot rate is high. The PW and bubble-free yield curves match better at high rates because the boundary behavior then has less of an effect. As in the PW model, the yield curve is humped as a function of maturity when $\kappa < 0$, but for realistic parameters, the peak occurs at longer maturities than in their model.[9]

Unlike the PW model, bubble-free yields are not monotonic in the short rate, but have a U-shape, and the lower bound is not zero. For the τ-period yield, the lower bound is

$$\underline{Y}^{\mathrm{BF}}(\tau) = \frac{B(\tau)}{\tau\xi(\tau)}\ell\mathrm{n}\,\frac{B(\tau)+\xi(\tau)}{B(\tau)} - \frac{1}{\tau}\ell\mathrm{n}\,\frac{\xi(\tau)}{B(\tau)+\xi(\tau)} > 0, \tag{102.12}$$

which is achieved when the interest rate is

$$r_{\min}(\tau) = \frac{1}{\xi(\tau)}\ell\mathrm{n}\,\frac{B(\tau)+\xi(\tau)}{B(\tau)}. \tag{102.13}$$

Figure 102.3 plots the yields for various maturities as a function of the spot rate. For interest rates above 4%, yields of all maturities through 20 years are nearly proportional to the spot rate just as they are (exactly) in the PW model. Below a spot rate of about 4%, yields have a strong U-shape in the spot rate. The minimum yield for each maturity is not zero, and for longer maturities can be quite high.

A side-effect of the bubble-free structure is that it restores the property that the values of zero-coupon bonds go to zero as τ goes to infinity. This can be verified by Equation (102.10) but it must be true since yields for all positive maturities are bounded away from zero, and $P(r,\tau) \leq \exp[-\underline{Y}(\tau)\tau]$.

Perhaps a more surprising result is that yields of all maturities become unboundedly large as the interest rate itself

[8] This condition is sufficient because zero is inaccessible for the CIR process $dr = k(\bar{r} - r)dt + \sigma\sqrt{r}d\omega$ if $2k\bar{r} \geq \sigma^2$, and by assumption, the specified process with drift $\mu(\cdot) \geq \frac{1}{2}\sigma^2$ for small r dominates the CIR process for sufficiently small r. This model is a special case of Heston, et al. (2007).

[9] In Figs. 102.2 through 102.6, the parameters used, $\kappa = -0.03$, $\sigma = 0.04$, are the midpoints of the estimates by PW, though they were fitting their bond pricing function not the bubble-free function.

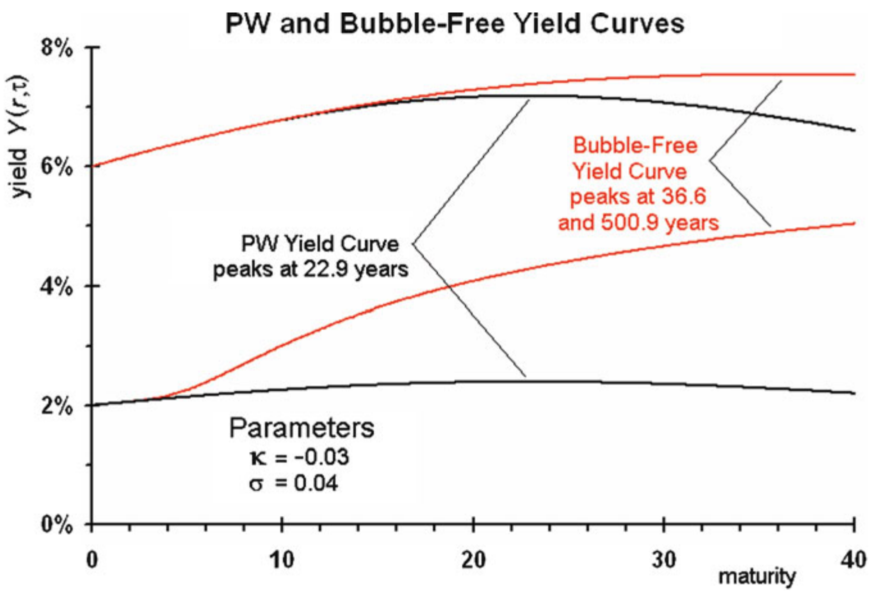

Fig. 102.2 Pan–Wu and bubble-free yield curves. The PW yield is given in Equation (102.6) as $-rB(\tau)/\tau$ with $B(\tau) \equiv 2(1-e^{-\gamma\tau})/[2\gamma + (\kappa-\gamma)(1-e^{-\gamma\tau})]$. The bubble free yield is $\tau^{-1}[B(\tau)r - \ell n(1-\exp[-r\xi(\tau)]]$ with $\xi(\tau) \equiv 2\gamma^2\sigma^{-2}[\gamma\sinh(\gamma\tau) + \kappa\cosh(\gamma\tau) - \kappa]^{-1}$ as given in Equation (102.11). In each case the risk-neutral evolution of the interest rate is $dr \mathrel{\hat{=}} -\kappa r dt + \sigma\sqrt{r}\cdot d\omega$. The parameter choices $\kappa = -0.03$ and $\sigma = 0.04$ correspond to the middle of the range estimated by Pan and Wu by fitting yield curves

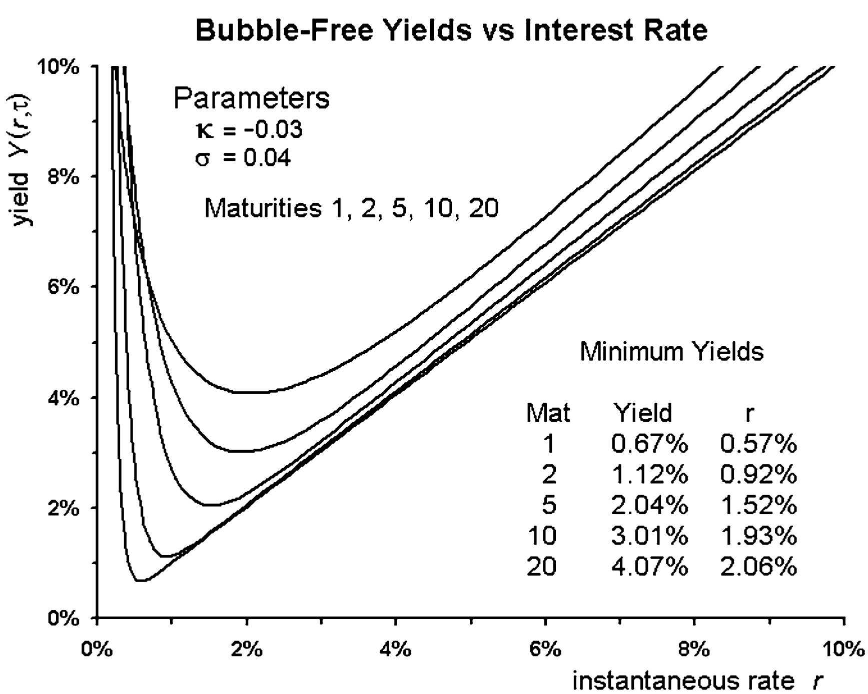

Fig. 102.3 Bubble-free yield as a function of the spot rate. The bubble free yield is given in Equation (102.11): $\tau^{-1}[B(\tau)r - \ell n(1-\exp[-r\xi(\tau)]]$ with $\xi(\tau) \equiv 2\gamma^2\sigma^{-2}[\gamma\sinh(\gamma\tau) + \kappa\cosh(\gamma\tau) - \kappa]^{-1}$. The parameter choices $\kappa = -0.03$ and $\sigma = 0.04$ correspond to the middle of the range estimated by Pan and Wu by fitting yield curves. All yields through 20 years are approximately proportional to the spot rate when the latter exceeds 4%. The lower bound for the τ-period yield is given in Equation (102.12) as $\underline{Y}(\tau) = B(\tau)/[\tau\xi(\tau)]\,(\ell n[B(\tau)+\xi(\tau)] - \ell n\,B(\tau)) - (\ell n\xi(\tau) - \ell n[B(\tau)+\xi(\tau)])/\tau$, which is achieved at a spot rate of $\ell n\,[1+\xi(\tau)/B(\tau)]/\xi(\tau)$

approaches zero. The intuition for this surprising result is found in the bond's risk premium. The bond's risk premium, $\pi(r, \tau)$, can be determined by Ito's Lemma

$$[r + \pi(r,\tau)]dt \equiv \frac{\mathbb{E}[dP]}{P} = \frac{\frac{1}{2}\sigma^2 r P_{rr} + \mu(r)P_r - P_\tau}{P}dt. \tag{102.14}$$

Comparing Equation (102.14) to the pricing equation, which uses the risk-neutral process, we have for the risk-neutral and true dynamics given in Equation (102.7)

$$\pi(r,\tau) = \frac{\partial P/\partial r}{P}[\mu(r) + \kappa r]. \tag{102.15}$$

The semi-elasticity, $(\partial P/\partial r)/P$, also determines each bond's return risk. Again by Ito's Lemma

$$\frac{dP}{P} - \mathbb{E}\left[\frac{dP}{P}\right] = \frac{\partial P/\partial r}{P}\sigma\sqrt{r}\,d\omega. \tag{102.16}$$

This, of course, is no coincidence. The absence of arbitrage requires that assets whose returns are perfectly correlated have risk premiums proportional to their standard deviations. The relations Equation (102.15) and (102.16) are true for both the PW and BF prices, though the semi-elasticities differ. Under the PW and bubble-free solutions the semi-elasticities are

$$\eta(r,\tau) \equiv \frac{\partial P/\partial r}{P} = \begin{cases} -B(\tau) & \text{PW} \\ -B(\tau) + \xi(\tau)\{\exp[r\xi(\tau)] - 1\}^{-1} & \text{BF} \end{cases} \tag{102.17}$$

Figure 102.4 shows the semi-elasticity function, $\eta(r,\tau) \equiv P_r/P$, for the bubble-free prices. (In the PW model, the semi-elasticity is constant as a function of r at the asymptote shown.) As the interest rate increases, the semi-elasticity approaches $-B(\tau)$, and both the risk and term premium are approximately independent of the interest rate. Where this flattening occurs depends on the bond's maturity. For bonds with maturities less than 2 years, η is nearly constant for all interest rates above 1.5%. For bonds with maturities in excess of 10 years, η does not flatten out until the interest rate is above 5%. This leads to the unbounded risk premiums and a number of other unusual effects.

Under the PW formula, the risk premium at low interest rates is negative since at $r = 0$ the premium is $\pi(0,\tau) = -B(\tau)\mu(0) < 0$. Because the bond price can never exceed one and, for the PW model, the price approaches one as r nears zero, the expected change in the bond price, and therefore its risk premium, must be negative near $r = 0$. For the bubble-free price, the semi-elasticity and therefore the risk premium becomes unboundedly large driving the bond price to zero as r approaches zero.

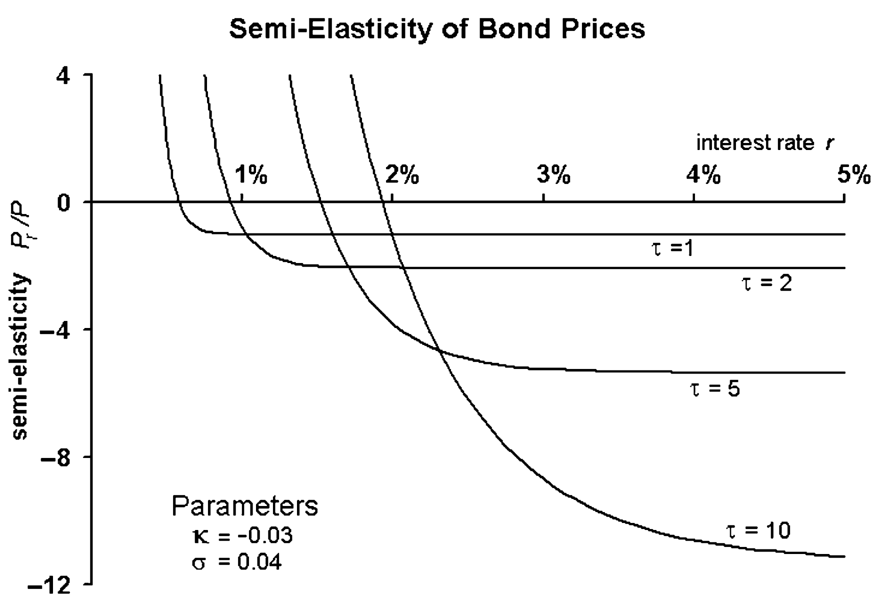

Fig. 102.4 Semi-elasticity of bubble-free bond prices. The semi-elasticity is given in Equation (102.17): $\eta(r,\tau) \equiv P_r/P = -B(\tau) + \xi(\tau)\{\exp[r\xi(\tau)] - 1\}^{-1}$ with $B(\tau) \equiv 2(1 - e^{-\gamma\tau})/[2\gamma + (\kappa - \gamma)(1-e^{-\gamma\tau})]$ and $\xi(\tau) \equiv 2\gamma^2\sigma^{-2}[\gamma\sinh(\gamma\tau)+\kappa\cosh(\gamma\tau)-\kappa]^{-1}$. The parameter choices $\kappa = -0.03$ and $\sigma = 0.04$ correspond to the middle of the range estimated by Pan and Wu by fitting yield curves. Any bond's return standard deviation is $|\eta(r,\tau)|\sigma\sqrt{r}$ with a negative value of η indicating that bond's price decreases with an increase in the interest rate. Any bond's term premium is $\eta(r,\tau)[\mu(r) + \kappa r]$

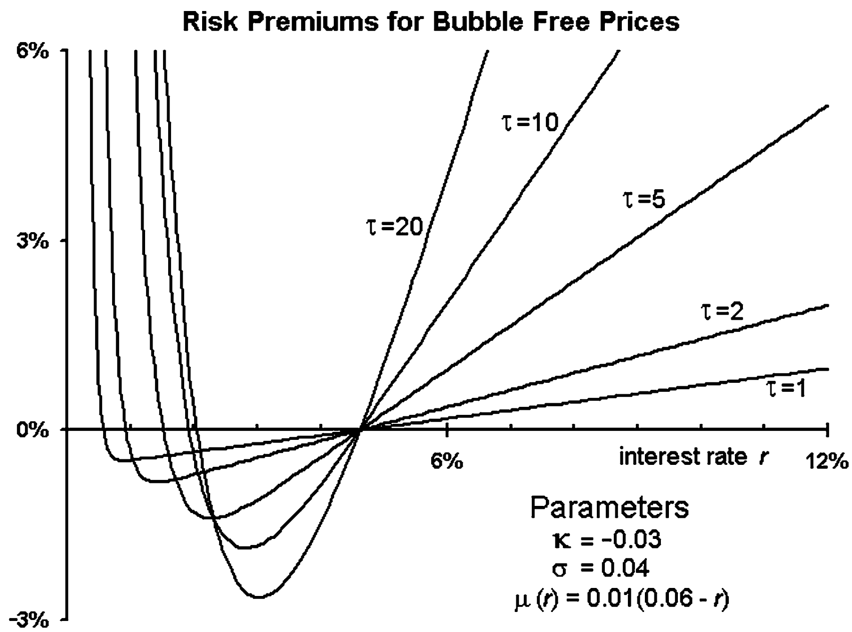

Fig. 102.5 Risk premiums of bubble-free bond prices. The risk premium is the product of the semi-elasticity in Equation (102.17): $\eta(r,\tau) \equiv P_r/P = -B(\tau) + \xi(\tau)\{\exp[r\xi(\tau)] - 1\}^{-1}$ and the difference between the true and risk-neutral expected changes, $\mu(r) + \kappa r$. It is plotted for $\mu(r) = 0.01(0.06 - r)$. The other parameter choices $\kappa = -0.03$ and $\sigma = 0.04$ correspond to the middle of the range estimated by Pan and Wu by fitting yield curves

At some interest rate levels, longer maturity bonds are less risky and have smaller term premiums than shorter maturity bonds. For example, when the spot rate is 2%, 10-year bonds are less risky than 2-year bonds. At low enough interest rates levels, the risk changes sign and bond prices are increasing in the interest rate. For example, this reversal occurs around 1.5% for 5-year bonds.

The change in sign of the semi-elasticity means that the risk premiums also change in sign. In fact in many cases the term premium changes in sign twice. This is true for example for all zero-coupon bonds if the true process mean is $\mu(r) = k(\bar{r} - r)$ with $k > \kappa$. Typical term premiums are illustrated in Fig. 102.5.

Both the PW and BF prices are self-consistent. That is, if zero coupon bonds traded at either model's prices, they would move in response to changes in interest rates just as the model predicts, and the risk premiums indicated in Equations (102.15) and (102.17) would be earned. However, this is always true of prices with bubbles. It is only in comparison to an alternate price that the bubble is obvious. The magnitude of the PW bubble is

$$P^{PW} - P^{BF} = e^{-r[B(\tau)+\xi(\tau)]}. \tag{102.18}$$

Unlike most bubbles, this bubble disappears at a fixed point in time, when the bond matures and is always between 0 and 1 in magnitude. Therefore, this bubble is finite in duration and bounded in size. If bonds sold at the higher PW price, an arbitrage would exist by selling the bonds and replicating them at a cost equal to this lower value according to the replicating hedge inherent in the partial differential equation.

Suppose bonds sold at the higher PW price. An investment matching the lower BF price could be achieved via a portfolio that holds $n(r,\tau) = P_r^{BF}/P_r^{PW}$ bonds and invests the residual, $P^{BF} - nP^{PW}$, in instantaneous lending. The change in value of this portfolio is

$$\begin{aligned}
d\text{Port} &= n(r,\tau)dP^{\text{PW}} + [P^{\text{BF}} - n(r,\tau)P^{\text{PW}}]r \cdot dt \\
&= n(r,\tau)\left([r + \eta^{\text{PW}}\mu(r)]P^{\text{PW}}dt + P_r^{\text{PW}}\sigma\sqrt{r}d\omega\right) \\
&\quad +[P^{\text{BF}} - n(r,\tau)P^{\text{PW}}]r \cdot dt \\
&= \frac{P_r^{\text{BF}}}{P_r^{\text{PW}}}\frac{P_r^{\text{PW}}}{P^{\text{PW}}}P^{\text{PW}}\mu(r)dt + P^{\text{BF}}r \cdot dt \\
&\quad +\frac{P_r^{\text{BF}}}{P_r^{\text{PW}}}P_r^{\text{PW}}\sigma\sqrt{r}d\omega \\
&= \left[r + \eta^{\text{BF}}(r,\tau)\mu(r)\right]P^{\text{BF}}dt + P_r^{\text{BF}}\sigma\sqrt{r}d\omega = dP^{\text{BF}}
\end{aligned} \tag{102.19}$$

which exactly matches the change in the BF formula, so this portfolio would always be self-financing and always equal in value to P^{BF}. A long position in this portfolio and a short position in the bond (at the PW price) would be an arbitrage. It would have a negative cost and be guaranteed to

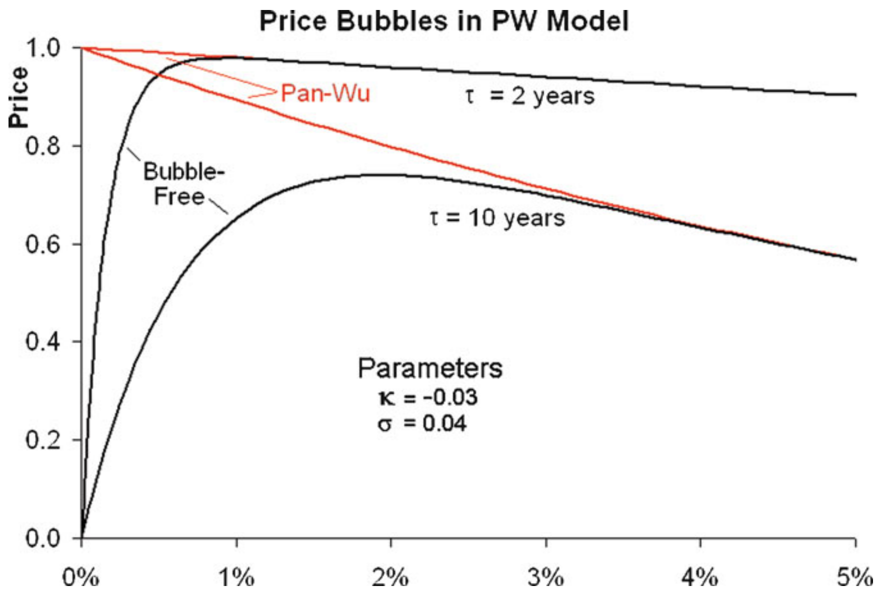

Fig. 102.6 Price bubble in Pan–Wu bond prices. The difference between the Pan–Wu and Bubble-Free price is a price bubble. The magnitude of the price bubble is $e^{-r[B(\tau)+\xi(\tau)]}$ where $B(\tau) \equiv 2(1-e^{-\gamma\tau})/[2\gamma+(\kappa-\gamma)(1-e^{-\gamma\tau})]$ and $\xi(\tau) \equiv 2\gamma^2\sigma^{-2}[\gamma\sinh(\gamma\tau) + \kappa\cosh(\gamma\tau)-\kappa]^{-1}$. Unlike most price bubbles this bubble is bounded in value between 0 and 1 and disappears after a fixed finite

be worth zero when the bond matured. Furthermore, since $P^{\text{PW}} > P^{\text{BF}}$, the arbitrage will never have a negative value at any point during the arbitrage.

The PW model is valid and gives a lower bound of zero for all yields only when zero is an absorbing barrier under *both* the true and risk-neutral interest rate processes. The next obvious question is can we have a lower bound of zero for all yields even when the interest rate process does display mean reversion? The answer is yes as we show in the next two sections of this paper.

102.5 Multivariate Affine Term-Structure Models with Zero Bounds on Yields

Pan and Wu (2006) claim: "Positivity and continuity, combined with no arbitrage, result in only one functional form for the term structure with three sources of risk."[10] But this claim is not true. There are other continuous term structure models in which there is no arbitrage and all interest rates are bounded by zero exactly. In fact there are infinitely many other affine term structure models with a single (excluding parameter) source of risk that have a lower bound of zero for yields.

The simplest such model extends the CIR dynamics so that the mean interest rate level, θ, is no longer a constant but is a weighted average of past interest rates. Specifically let x_t be an exponentially smoothed average of past spot rates with an average lag of $1/\delta$[11]

$$x_t = \delta\int_0^\infty e^{-\delta s} r_{t-s} ds. \tag{102.20}$$

The dynamics of x_t are locally deterministic, and the evolution of the state space is

$$\begin{aligned} dr_t &= \kappa(x_t - r_t)dt + \sigma\sqrt{r_t}d\omega_t \\ dx_t &= \delta(r_t - x_t)dt \end{aligned} \tag{102.21}$$

with a single source of risk, $d\omega$.

[10] Pan and Wu refer to their model as a three-factor (i.e., r, κ, σ) model with a single dynamic factor, r. In fitting their model they allow the parameters to vary over time, hence adding two additional sources of risk. This "stochastic-parameter" method has been widely used in practice since being introduced to term-structure modeling by Black et al. (1990). As Pan and Wu point out, this is inconsistent with their derivation, which assumes the parameters to be constant. Were the parameters actually varying, then bond prices would not be given by Equation (102.5) or (102.10). A true multifactor model giving results similar to PW would be a special case of the Longstaff and Schwartz (1992) multifactor extension to the CIR model with the constants in the drift terms set to zero. That is, $ds_i = -\kappa_i s_i dt + \sigma_i\sqrt{s_i}d\omega_i$ and $r = s_1 + s_2 + s_3$. The zero-coupon yield to maturity in this model is $Y(\mathbf{s},\tau) = \tau^{-1}\sum B_i(\tau)s_i$. This Longstaff–Schwartz model is an immediate counterexample to PW's claim that their formula is unique, but as in the PW model each of the state variables can be trapped at zero, and the interest rate becomes trapped at zero once all three state variables are so trapped.

[11] The average lag in the exponential average is $\delta\int_0^\infty se^{-\delta s}ds = \delta^{-1}$. An exponentially smoothed average is the continuous-time equivalent of a discrete-time geometrically smoothed average $x_t = (1-\eta)\sum\eta^s r_{t-s}$. Geometrically smoothed averages were first suggested in interest rate modeling by Malkiel (1966).

Unlike in the PW model, the interest rate cannot be trapped at zero under the process in Equation (102.21). Since x is a positively weighted average of past values of r, it clearly must remain positive. Even were r to reach zero,[12] x would only decay towards zero at the rate δ. But when r reaches zero, its diffusion term is zero so the immediate change in r is $dr = \kappa x \cdot dt$, and as x is still positive at this point, $dr > 0$, r immediately becomes positive again.

Assuming the factor risk premium is linear in the state variables, $\pi(r, x, \tau) = (\psi_0 r + \psi_1 x) P_r / P$, then the bond-pricing equation is[13]

$$0 = \frac{1}{2}\sigma^2 r P_{rr} + [(\kappa - \psi_1)x - (\kappa + \psi_0)r] P_r + \delta(r - x) P_x - rP - P_\tau. \tag{102.22}$$

The price of a zero-coupon bond has the form

$$P(r, x, \tau) = \exp[-b(\tau)r - c(\tau)x]. \tag{102.23}$$

This pricing formula can be easily verified by substituting the partial derivatives of P in Equation (102.23) into the pricing Equation (102.22). The terms proportional to r and x must separately sum to zero so the functions b and c are the solutions to the linked ordinary first-order differential equations

$$\begin{aligned} r \text{ terms}: 0 &= \tfrac{1}{2}\sigma^2 b^2(\tau) + (\kappa + \psi_0) b(\tau) - \delta c(\tau) - 1 + b'(\tau) \\ &\quad \text{with} \quad b(0) = 0 \\ x \text{ terms}: 0 &= (\psi_1 - \kappa) b(\tau) + \delta c(\tau) + c'(\tau) \\ &\quad \text{with} \quad c(0) = 0. \end{aligned} \tag{102.24}$$

While a closed-form solution to Equation (102.24) is not known, $b(\tau)$ and $c(\tau)$ can be easily computed by numerically integrating the two equations.[14] Note that this calculation need only be done once for a given set of parameters; bond prices at each interest rate level and maturity need not be separately computed as in a finite difference or binomial model.

For this model, the yield to maturity on a τ-period zero-coupon bond is

$$Y(r, x, \tau) \equiv -\frac{1}{\tau}\ell\text{n}\, P(r, \tau) = \frac{b(\tau)}{\tau} r + \frac{c(\tau)}{\tau} x. \tag{102.25}$$

Since both r and x can be arbitrarily close to zero, yields to maturity have lower bounds of exactly zero for all maturities as in the PW model even though the interest rate exhibits mean reversion as in the CIR model. It should be noted, however, that the lower bound of zero cannot be approached immediately. In the PW model, yields are proportional to the interest rate, and since r can be arbitrarily close to zero at any time, so can the yields. In this two factor model r can also be arbitrarily close to zero at any time, but x_t cannot be smaller than $x_0 e^{-\delta t}$; therefore, the τ-period yield cannot be less than $\tau^{-1} c(\tau) x_0 e^{-\delta t}$ at time t.

Figures 102.7 and 102.8 display the yield curve for the two-factor model. Figure 102.7 compares it to the CIR yield curve for the same parameters and when $x_t = \theta$. The two-factor yield curve is less steeply sloped than the CIR yield curve because x and r both tend to move towards each other rather than r simply moving toward θ. This lessens future expected movements in the interest rate. Figure 102.8 illustrates the effect on the yield curve of different values of δ. The larger is δ, the less steeply sloped is the yield curve as x then moves more strongly towards r.

Both figures also show that the long zero-coupon yields go to zero as the maturity lengthens regardless of the parameter values (provided δ is not zero which is the CIR model). This is true even though the interest rate can never be trapped at zero as in the PW model verifying that the latter is not a prerequisite to the former.

One problem with this model, apart from the lack of a simple closed-form expression for the solution, is that the long-term behavior of the interest rate is unrealistic. The model displays short-term mean reversion with r staying near x, but over long periods of time, the interest rate is likely to become

[12] It is irrelevant for this discussion whether or not zero is accessible; if zero is not accessible for r then clearly neither r nor x can become negative. By comparison to the CIR process, however, we can determine that 0 is accessible for r (though not x). Specifically compare the CIR process with $\theta < 2\sigma^2/\kappa$ to the process in Equation (102.21). The diffusion terms are identical and the excepted change under the bivariate process is smaller than for the CIR process whenever both r and x are less than θ. Since 0 is accessible for the CIR process, it must be accessible for the dominated bivariate process. The fact that r and x can be larger than θ does not alter this conclusion as the accessibility of 0 depends only on the behavior of r and x near 0.

[13] Only r is locally stochastic, so the risk premium is proportional to P_r/P and independent of P_x/P. The risk-neutral and true processes are equivalent if and only if $\psi_1 < \kappa$ so that r remains positive under the risk-neutral process as well.

[14] In particular,

$$\begin{aligned} b(\tau + \Delta\tau) &\approx b(\tau) \\ &\quad - \left[\tfrac{1}{2}\sigma^2 b^2(\tau) + (\kappa + \psi_0) b(\tau) - \delta c(\tau) - 1\right] \Delta\tau \qquad b(0) = 0 \\ c(\tau + \Delta\tau) &\approx c(\tau) - [(\psi_1 - \kappa) b(\tau) + \delta c(\tau)] \Delta\tau \qquad c(0) = 0. \end{aligned}$$

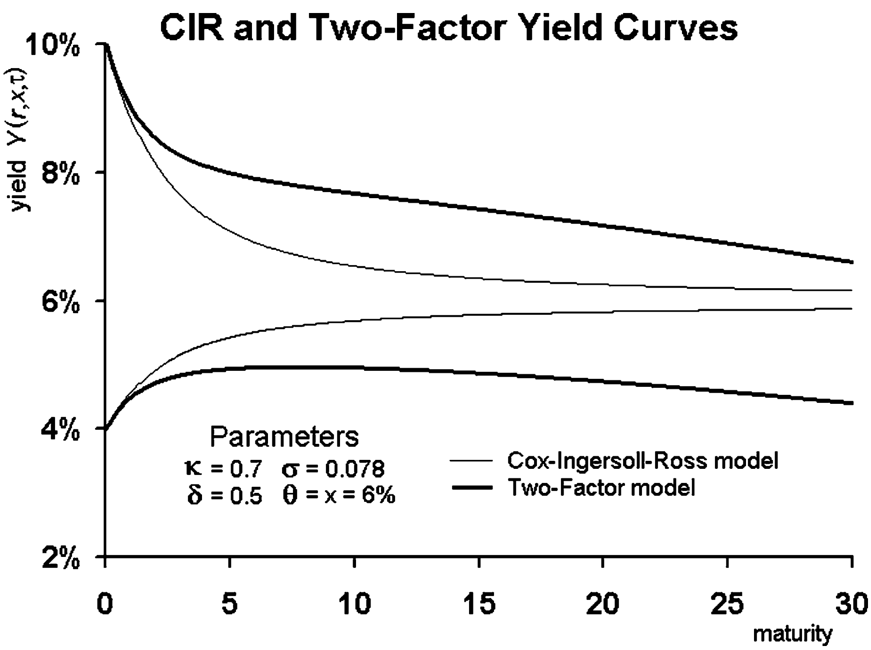

Fig. 102.7 Cox–Ingersoll–Ross and two-factor affine yield curves. This figure shows the yields to maturity for the CIR and two-factor model in Equation (102.25). The parameters are $\kappa = 0.7$, $\delta = 0.5$, $\sigma = 0.078$. The two factor yield curve is less steeply sloped than the CIR yield curve and is always downward sloping to zero for large maturities

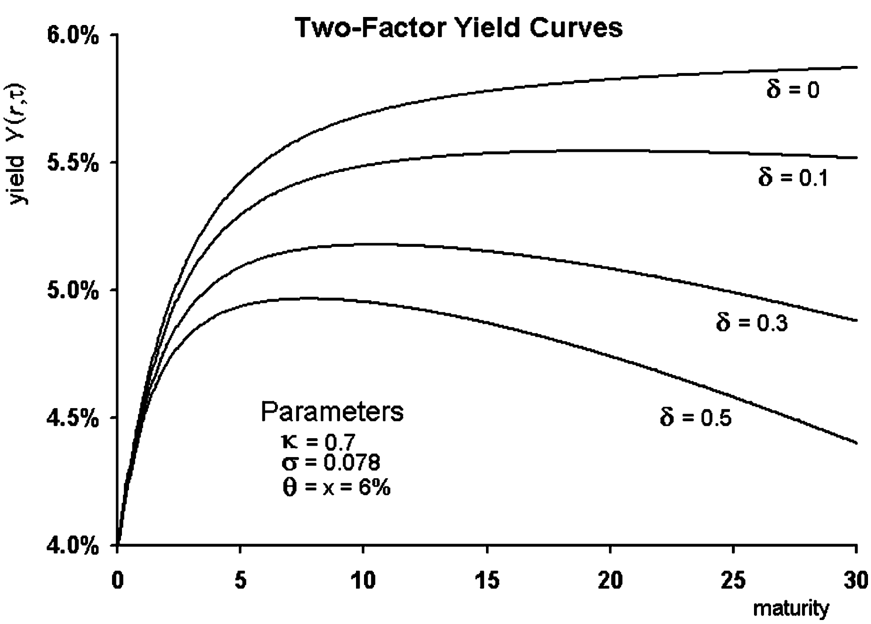

Fig. 102.8 Two-factor yield curves illustrating dependence on δ. This figure shows the yields to maturity for the two-factor model in Equation (102.25). The parameters are $\kappa = 0.7$ and $\sigma = 0.078$. The yield curve is plotted for various values of δ, the reciprocal of the average lag in the central tendency mean

very large as there is no steady state distribution for the interest rate. In particular, the expected interest rate at time t,

$$\mathbb{E}[r_t|r_0,x_0] = r_0 - \frac{\kappa}{\kappa+\delta}(r_0-x_0)\left(1-e^{-(\kappa+\delta)t}\right) \underset{t\to\infty}{\longrightarrow} \frac{\delta r_0+\kappa x_0}{\delta+\kappa}, \tag{102.26}$$

does not reach a limit independent of the current state, and the variance of r_t

$$\mathrm{Var}[r_t^2] \sim \frac{\sigma^2\delta^2}{2\kappa(\kappa+\delta)^3}(\delta r_0+\kappa x_0)t + o(t). \tag{102.27}$$

becomes unbounded. So, in the limit, the distribution of r_t is completely diffuse over all positive values. Of course, a similar criticism can also be leveled against the PW model. Those dynamics also have no steady state. The interest rate variance becomes unbounded, and the spot rate is either trapped at zero or has an infinite expectation.[15]

Also as in the PW model, the prices of very long zero-coupon bonds are bounded away from zero as τ becomes large. For given values of r and x, the smallest that a bond price can be is

$$P(r,x,\tau) > \underline{P} \equiv \exp[-b_\infty r - c_\infty x] \qquad \forall \tau$$
$$\text{where} \qquad b_\infty \equiv \frac{\sqrt{(\psi_0+\psi_1)^2+2\sigma^2}-\psi_0-\psi_1}{\sigma^2}$$
$$c_\infty \equiv \frac{\kappa-\psi_1}{\delta} b_\infty. \tag{102.28}$$

This can be verified with Equation (102.24). The derivatives c' and b' are positive (zero) whenever c is less than (equal to) $(\kappa-\psi_1)b/\delta$ and δc is greater than (equal to) $1-(\kappa+\psi_0)b-\frac{1}{2}\sigma^2 b^2$, respectively. Since $b(0)=c(0)=c'(0)=0$ and $b'(0)>0$, both b and c increase monotonically staying in the range bounded by the $b'=0$ and $c'=0$ curves approaching b_∞ and c_∞ as $\tau\to\infty$.

Similar properties are true for all affine models with multiple state variables. Consider the general multiple state-variable extension to the CIR model with a risk-neutral evolution of

$$\begin{aligned} dr &\triangleq \hat{\mu}(r,\mathbf{x})dt + \sigma\sqrt{r}\,d\omega \\ &\triangleq \left(\hat{K}+\hat{\kappa}_0 r + \sum \hat{\kappa}_i x_i\right)dt + \sigma\sqrt{r}\,d\omega \\ dx_j &= \mu_j(r,\mathbf{x})dt = \left(\Delta_j + \delta_{j0} r + \sum \delta_{ji} x_i\right)dt\, j = 1,\dots I. \end{aligned} \tag{102.29}$$

The same analysis leading to Equation (102.24) shows the yield curve will be affine

$$Y(r,\mathbf{x},\tau) = a(\tau) + b(\tau)r + \sum c_i(\tau)x_i. \tag{102.30}$$

so all yields will have a lower bound of exactly zero only if $a(\tau)\equiv 0$ and all state variables remain nonnegative. The former requires that the constant terms, $\hat{K}$ and all Δ_j, are zero. The latter requires additionally that all $\hat{\kappa}_i(i\neq 0)$ and all $\delta_{ij}(j\neq i)$ are nonnegative and that all state variables are nonnegative initially.[16] The interest rate will never be trapped at zero so long as one of the parameters, $\hat{\kappa}_i(i\neq 0)$, is positive. However, as in the two-variable case, there will not be a finite-variance steady-state distribution, and the prices of all zero-coupon bonds will be bounded away from zero even as the maturity grows without bound.

Our search for an affine model in which the interest rate is well behaved in the long term and yields of all maturities are bounded below exactly by zero has not been successful. But such models are possible outside of the affine structure. One such model has already appeared in the literature. It is the three-halves power model used in the two-factor CIR model. This model is discussed in the next section.

102.6 Non-Affine Term Structures with Yields Bounded at Zero

Outside of the affine class, many models of the term structure will have zero lower bounds for all rates and still be well-behaved with mean reversion and a finite-variance steady-state distribution with no atom of probability at $r=0$. One such model that admits to a closed form solution for bond prices is the three-halves power model in which the interest rate's evolution is

$$dr_t = \kappa r_t(\theta - r_t)dt + \sigma r_t^{3/2} d\omega_t, \tag{102.31}$$

with $\kappa,\theta,\sigma>0$.[17] This structure was introduced in the original CIR (1985) paper to model the rate of inflation. Like the CIR process, this diffusion displays mean reversion with a central tendency of θ, and the local variance vanishes when $r_t=0$ so negative rates are impossible.[18] The interest rate has a finite-variance steady state distribution with a density function, mean, and variance of

---

[15] If $\kappa\geq 0$ in the PW model, then the interest rate is eventually trapped at zero with probability one. If $\kappa<0$, then the expected interest rate and variance become infinite, $\mathbb{E}[r_t|r_0]=r_0e^{-\kappa t}\to\infty$, $\mathrm{Var}[r_t|r_0]=r_0\sigma^2\left(e^{-\kappa t}-e^{-2\kappa t}\right)/\kappa\to\infty$, and there is an atom of probability for $r_\infty=0$ equal to $\exp(2\kappa r_0/\sigma^2)$.

[16] In addition, the risk-neutral process must also be equivalent to the true process so all state variables must remain nonnegative under the latter as well.

[17] The central tendency parameter, θ, and the adjustment parameter, κ, can be zero just as in the PW model. For the three-halves process, the origin remains inaccessible even in these cases, and yields are still bounded below by zero. There is, however, no finite-variance steady-state distribution.

[18] Since the drift term is zero at an interest rate of zero, $r_t=0$ is technically an absorbing state. However, zero is inaccessible for all parameter values so r is never trapped there. To verify this define $z=1/r$. Then using Itô's Lemma, the evolution of z is $dz=(\kappa+\sigma^2-\kappa\theta z)dt-\sigma\sqrt{z}\,d\omega$. Since $z=\infty$ is inaccessible for the square root process with linear drift, zero is inaccessible for $r=1/z$. Note also that $2(\kappa+\sigma^2)>\sigma^2$ so zero is inaccessible for z guaranteeing that ∞ is inaccessible for r in Equation (102.31).

$$f(r,\infty) = \frac{[(\beta-1)\theta]^{1+\beta}}{\Gamma(\beta+1)} r^{-\beta} e^{-(\beta-1)\theta/r}$$
$$\bar{r}_\infty \equiv \mathbb{E}[r_\infty] = \frac{\beta-1}{\beta}\theta \qquad \text{Var}\,[r_\infty] = \frac{\sigma^2}{2\kappa}\frac{(\beta-1)^2\theta^2}{\beta^2} = \frac{\sigma^2}{2\kappa}\bar{r}^2_\infty$$
$$\text{where} \qquad \beta \equiv 1 + 2\kappa/\sigma^2. \tag{102.32}$$

The three-halves power model may fit the data better than the CIR model and other proposed models. Chan et al. (1992) tested the general specification for interest rate evolution

$$r_t = \alpha + \beta r_{t-1} + \varepsilon_t \qquad \mathbb{E}[\varepsilon_t] = 0 \quad \mathbb{E}[\varepsilon_t^2] = \sigma^2 r_t^{2\gamma}. \tag{102.33}$$

Using GMM they found a best unrestricted fit of $\gamma = 1.4999$ with a standard error of 0.252. This, of course, is not a direct test of Equation (102.31), which has a quadratic rather than linear form for the expected interest rate, but the estimated value of γ is almost exactly what this model calls for. The CIR-square-root, Merton (1975, 1990), and Vasicek (1977) models are all well outside the usual confidence intervals, and the Brennan and Schwartz (1982) and Dothan (1978) models are just at the 5% significance level.[19]

We assume that the risk premium is of the form $\pi(r,\tau) - r = (\psi_1 r + \psi_2 r^2)P_r/P$ so that the risk-neutral process also has the form in Equation (102.31). The risk-neutral drift is then $\hat{\kappa} r(\hat{\theta} - r)$ with $\hat{\kappa} \equiv \kappa + \psi_2$ and $\hat{\theta} \equiv (\kappa\theta - \psi_1)/(\kappa + \psi_2)$.[20] Zero-coupon bond prices for this model can be determined from Cox et al. (1985) as

$$P(r,\tau) = \frac{\Gamma(\nu-\delta)}{\Gamma(\nu)}[c(\tau)/r]^\delta M(\delta,\nu,-c(\tau)/r)$$
$$\text{where} \qquad c(\tau) \equiv 2\frac{\kappa\theta-\psi_1}{\sigma^2}\left[e^{(\kappa\theta-\psi_1)\tau}-1\right]^{-1}, \qquad \hat{\beta} \equiv 1 + 2(\kappa+\psi_2)/\sigma^2$$
$$\delta \equiv \tfrac{1}{2}(\hat{\beta}^2 + 8/\sigma^2)^{1/2} - \tfrac{1}{2}\hat{\beta}, \qquad \nu \equiv 1 + (\hat{\beta}^2 + 8/\sigma^2)^{1/2}, \tag{102.34}$$

$\Gamma(\cdot)$ is the gamma function, and $M(\cdot)$ is the confluent hypergeometric function.[21]

The asymptotic long rate is

$$Y_\infty \equiv \lim_{\tau\to\infty}\left[-\tfrac{1}{\tau}\ell\text{n}\, P(r,\tau)\right] \sim \begin{cases} \frac{\delta}{\tau}\ell\text{n}\left(e^{(\kappa\theta-\psi_1)\tau}-1\right) + O(\tau^{-1}) & \xrightarrow[\tau\to\infty]{} \quad (\kappa\theta-\psi_1)\delta = \hat{\kappa}\hat{\theta}\delta \quad \text{for} \quad \hat{\kappa} > 0 \\ \frac{\delta}{\tau}o(\tau) & \xrightarrow[\tau\to\infty]{} \quad 0 \quad \text{for} \quad \hat{\kappa} \le 0. \end{cases} \tag{102.35}$$

[19] The volatility parameter is $\gamma = 0$ for Merton (1990) and Vasicek (1977), $\gamma = \frac{1}{2}$ for CIR, and $\gamma = 1$ for Brennan and Schwartz (1982), Dothan (1978) and Merton (1975). Each of these models with the exception of Merton's (1975) does have a linear form for the expected change in r.

[20] We require that $\psi_2 \ge -(\kappa + \frac{1}{2}\sigma^2)$ so that the true and risk-neutral processes are equivalent. If this condition is not satisfied then the risk-neutral process is explosive, and the interest rate can become infinite in finite time. As shown in footnote 18, the risk-neutral process for $z \equiv 1/r$ has $\hat{\mathbb{E}}[dz] = (\hat{\kappa} + \sigma^2 - \hat{\kappa}\hat{\theta}z)dt$. So if ψ_2 violates the condition given, $2(\hat{\kappa} + \sigma^2) < \sigma^2$, and 0 is accessible for z implying that ∞ is accessible in finite time for r under the risk-neutral (though not true) process.

[21] See Abramowitz and Stegum (1964) for the properties of the gamma and confluent hypergeometric functions.

When $\psi_2 < \kappa$ (i.e., $\hat{\kappa} > 0$), Y_∞ is a positive constant a bit less than the risk-neutral central tendency level $\hat{\theta}$.[22] The asymptotic bond price, $P(r, \infty)$, is also zero when $\psi_2 \le \kappa$.[23] For short maturities

$$Y(r,\tau) = r + \tfrac{1}{2}r[\kappa\theta - \psi_1 - (\kappa + \psi_2)r]\tau + O(\tau^2)$$
$$= r + \tfrac{1}{2}\hat{\kappa}r(\hat{\theta} - r)]\tau + O(\tau^2) \qquad (102.36)$$

so the yield curve is upward sloping at $\tau = 0$ whenever the interest rate is below the risk-neutral central tendency point, $\hat{\theta}$. The long rate is $Y_\infty = \hat{\kappa}\hat{\theta}\delta < \hat{\theta}$ since $\hat{\kappa}\delta < 1$; therefore, the yield curve is humped shape whenever the spot interest rate is between the values $\hat{\kappa}\delta\hat{\theta}$ and $\hat{\theta}$.

Even though the asymptotic yield to maturity is a positive constant (when $\hat{\kappa} > 0$), the yield to maturity on any finite maturity bond is bounded below exactly by zero. As shown in the Appendix, the τ-period yield to maturity is an analytic function of r and can be expressed as a power series with no leading constant term,

$$Y(r,\tau) \equiv -\frac{1}{\tau}\ell n\, P(r,\tau) = \frac{2r}{\sigma^2\tau c(\tau)} - \frac{2(\kappa+\psi_2)r^2}{\sigma^4\tau c^2(\tau)} + \cdots. \qquad (102.37)$$

Therefore, for any finite maturity, the yield to that maturity is less than $2r/[\sigma^2\tau c(\tau)]$ when r is sufficiently small and approaches its lower bound of zero as r does.[24]

These properties can also be verified for many models even when a closed-form solution to the bond pricing problem cannot be found. Suppose that the risk-neutral expected change and variance in the interest rate process and the yield to maturity are analytic functions of the interest rate at $r = 0$; that is, they can be expressed as the infinite power series

$$\sigma^2(r) = \sum_{i=0}^{\infty} s_i r^i \qquad \hat{\mu}(r) = \sum_{i=0}^{\infty} m_i r^i$$
$$Y(r,\tau) = \sum_{i=0}^{\infty} \tau^{-1} y_i(\tau) r^i. \qquad (102.38)$$

A yield to maturity will have the desired lower bound of exactly zero if and only if the lead term in its expansion is zero; *that is*, $y_0(\tau) \equiv 0$.

[22] Holding σ constant, $\hat{\kappa}\delta$ increases from 0 to 1 when $\hat{\beta}$ ranges from 1, its lowest value, to ∞.

[23] When $\psi_2 > \kappa$, the asymptotic bond price is $P(r,\infty) = \Gamma(\upsilon - \delta)/\Gamma(\upsilon)[-2\hat{\kappa}/\sigma^2]^\delta M(\delta, \upsilon, 2\hat{\kappa}/\sigma^2) > 0$.

[24] The lower bound for any yield is zero since for every finite τ, there exists an interest rate r_τ such that $Y(r,\tau) < \varepsilon$ for all $r < r_\tau$. The bound is not a uniform one for all τ, however, in that r_τ depends on τ. The bound cannot be uniform since the asymptotic long rate is a positive constant.

Substituting $P(r,\tau) = \exp[-\sum y_i(\tau)r^i]$ and its derivatives into the bond pricing equation gives

$$0 = \frac{1}{2}\sum s_i r^i \left(\left[\sum (i+1)y_{i+1}(\tau)r^i\right]^2 - \sum (i+1)(i+2)y_{i+2}(\tau)r^i\right)$$
$$- \sum m_i r^i \left[\sum (i+1)y_{i+1}(\tau)r^i\right] - r + \left[\sum y_i'(\tau)r^i\right]$$
$$= \frac{1}{2}s_0\left(y_1^2(\tau) - 2y_2(\tau)\right) - m_0 y_1(\tau) + y_0'(\tau)$$
$$+ r\cdot[\cdots] + r^2\cdot[\cdots] + \cdots \qquad (102.39)$$

The terms in each power of r must be identically zero, so $y_0(\tau)$ will be constant (and therefore 0 since $y_i(0) = 0$) if and only if $s_0 = m_0 = 0$. If $s_0 = 0$, then zero is a natural barrier for the interest rate process and negative rates will be precluded if $\mu(0) \ge 0$.[25] If m_0 also is zero, then $r_t = 0$ is an absorbing barrier; however, the barrier might be inaccessible as in the three-halves power model.

Therefore, models like CIR (1985) or Vasicek (1977) with a linear expected change in the interest rate, $\mu(r) = \kappa(\theta - r)$, will not have yields with a lower bound of zero. Conversely, a model like that in Merton (1975) with $\sigma(r) = vr$ and $\mu(r) = ar - br^2$ will have all yields bounded below by exactly zero.

102.7 Non-Zero Bounds for Yields

All of the results presented here apply to other constant bounds for yields that derive from restricted interest rate processes. Suppose the interest rate is restricted so that it can never go below $\underline{r}$. Under what conditions will all yields have this same lower bound of $\underline{r}$? This question can be easily answered by defining the modified interest rate, $\rho \equiv r - \underline{r}$. Since ρ and r are linearly related, the dynamics for ρ will be identical to those for r translated down by $\underline{r}$, and the modified rate, ρ, can never be negative. All yields will have an identical lower bound of $\underline{r}$ whenever yields in the modified economy are all bounded below by zero.

To verify this claim write the interest rate dynamics as $dr = \mu(r)dt + \sigma(r)d\omega$. Now express the bond prices in the original economy as $P(r,\tau) = e^{-\underline{r}\tau}\Theta(\rho,\tau)$. The bond pricing equation can then be reexpressed as

$$0 = \frac{1}{2}\sigma^2(r)P_{rr} + \mu(r)P_r - rP - P_\tau$$
$$= \frac{1}{2}\sigma^2(\rho + \underline{r})\Theta_{\rho\rho} + \mu(\rho + \underline{r})\Theta_\rho - \rho\Theta - \Theta_\tau \qquad (102.40)$$

[25] Zero can also be an inaccessible natural barrier for the process if $\mu(0) = \infty$ even if $\sigma(0) \ne 0$.

This is just the regular bond pricing equation for an interest rate ρ with expected change and standard deviation of $\mu_\rho(\rho) \equiv \mu(\rho + \underline{r})$ and $\sigma_\rho(\rho + \underline{r})$. So all yields in the original economy will be larger than those in the modified economy by exactly $\underline{r}$. In particular, all yields will have a lower bound of $\underline{r}$ if, and only if, they have a lower bound of 0 in the modified economy.

For example, in Sundaresan (1984),[26] the dynamics of the real interest rate are

$$dr = (\alpha - 1)(r + \sigma^2)\left[(r - \delta + \tfrac{1}{2}\alpha\sigma^2)dt + \sigma d\omega\right] \tag{102.41}$$

with $r > -\sigma^2$. Rewriting this in terms of $\rho \equiv r + \sigma^2$, we have

$$d\rho = (\alpha - 1)\rho\left[(\rho - \delta + \frac{1}{2}(\alpha - 2)\sigma^2)dt + \sigma d\omega\right]. \tag{102.42}$$

which is identical to Merton's (1975) model. All yields have a lower bound of 0 in Merton's model; therefore, all yields in Sundaresan's model are bounded below by exactly $-\sigma^2$.

102.8 Conclusion

This paper has established the properties of interest-rate models in which all yields have a lower bound equal to the lower bound of the interest rate itself. In particular, when the interest rate must remain positive, it is possible for all yields to have a lower bound of zero as well. Yields are not simple arithmetic expectations (or even risk-neutral expectations) of future short rates; therefore, they can be as low as the lowest possible interest rate even when the interest rate process displays mean reversion and has a finite-variance steady-state distribution.

This paper also illustrates the problems that can arise when then true and risk-neutral stochastic processes for the interest rate have different boundary behaviors. Price bubbles can be introduced unless events that are impossible under the true distribution are assigned zero probability under the risk-neutral process as well. In some cases, such as the Pan Wu model, alternate bubble-free prices can be derived.

[26] Equation (102.41) fixes a typo in the second unnumbered equation on p. 84 of Sundaresan (1984).

References

Abramowitz, M. and I. Stegum. 1964. *Handbook of mathematical functions*, Dover Publications, New York.

Black, F., E. Derman, and W. Toy. 1990. "A one-factor model of interest rates and its application to treasury bond options." *Financial Analysts Journal* 46, 33–39.

Brennan, M. J. and E. S Schwartz. 1982. "An equilibrium model of bond pricing and a test of market efficiency." *Journal of Financial and Quantitative Analysis* 41, 301–329.

Chan, K. C., G. Andrew Karolyi, F. Longstaff, and A. Sanders 1992. "An empirical comparison of alternative models of the short-term interest rate." *Journal of Finance* 47, 1209–1227.

Cox, J. C., J. E. Ingersoll, and S. A. Ross. 1981. "A re-examination of traditional hypotheses about the term structure of interest rates." *Journal of Finance* 36, 769–799.

Cox, J. C., J. E. Ingersoll, and S. A. Ross. 1985. "A theory of the term structure of interest rates." *Econometrica* 53, 385–408.

Dothan, U. 1978. "On the term structure of interest rates." *Journal of Financial Economics* 6, 59–69.

Dybvig, P. H., J. E. Ingersoll, and S. A. Ross. 1996. "Long forward and zero-coupon rates can never fall." *Journal of Business* 69, 1–25.

Heston, S. L., M. Lowenstein, and G. A. Willard. 2007. "Options and bubbles." *Review of Financial Studies* 20, 359–390.

Longstaff, F. A. 1992. "Multiple equilibria and term structure models." *Journal of Financial Economics* 32, 333–344.

Longstaff, F. A. and E. S. Schwartz. 1992. "Interest rate volatility and the term-structure of interest rates: a two-factor general equilibrium model." *Journal of Finance* 47, 1259–1282.

Malkiel, B. 1966. *The term structure of interest rates: expectations and behavior patterns*, Princeton University Press, Princeton.

Merton, R. C. 1975. "An asymptotic theory of growth under uncertainty." *Review of Economic Studies* 42, 375–393.

Merton, R. C. 1990. "A dynamic general equilibrium model of the asset market and its application to the pricing of the capital structure of the firm," chapter 11 in *Continuous-time finance* Basil Blackwell Cambridge, MA.

Pan, E. and L. Wu. 2006. "Taking positive interest rates seriously." chapter 14, *Advances in Quantitative Analysis of Finance and Accounting* 4, Reprinted as chapter 98 herein.

Sundaresan, M. 1984. "Consumption and equilibrium interest rates in stochastic production economies." *Journal of Finance* 39, 77–92.

Vasicek, O. 1977. "An equilibrium characterization of the term structure." *Journal of Financial Economics* 5, 177–188.

Appendix 102A

102A.1 Derivation of the Probability and State price for $r_T = 0$ for the PW Model

Let $H(r, t)$ be probability that $r_T = 0$ conditional on $r_t = r$ for the square root stochastic process with no mean reversion: $dr = -\kappa r dt + \sigma r^{1/2} d\omega$. Let $Q(r, \tau)$ be the state price for the state $r_T = 0$; that is, $Q(r, \tau)$ is the value at time t when the interest rate is r of receiving \$1 if the interest rate is zero at time $T = t + \tau$. This section of the appendix verifies Equations (102.8) and (102.9) in the text that

$$H(r,t;T) = \exp[-h(\tau)r] \qquad \text{where} \quad h(\tau) \equiv 2\kappa\big/\sigma^2(e^{\kappa\tau}-1)$$
$$Q(r,\tau) = \exp[q(\tau)r] \qquad \text{where} \quad q(\tau) \equiv [\kappa - \gamma\coth(\tfrac{1}{2}\gamma\tau)]\big/\sigma^2. \tag{102A.1}$$

The probability H satisfies the Kolmogorov backward equation and boundary conditions

$$0 = \tfrac{1}{2}\sigma^2 r H_{rr} - \kappa r H_r - H_\tau \qquad \text{subject to } H(0,t;T) = 1 \quad \text{and} \quad H(r,T;T) = 0. \tag{102A.2}$$

The condition $H(0,t) = 1$ must be satisfied because zero is an absorbing state so once r reaches zero before time T, it will be there at time T with probability one.

For the solution given, $H(0,\ t) = 1$ and $H(r,\ T) = 0$ are readily confirmed. The solution itself can be verified by differentiating and substituting into Equation (102A.2). The partial derivatives we need are

$$H_r = -\exp[-rh(\tau)]h(\tau) \qquad H_{rr} = \exp[-rh(\tau)]h^2(\tau) \qquad H_\tau = -\exp[-rh(\tau)]rh'(\tau)$$
$$\text{with} \quad h'(\tau) = -2\kappa^2\sigma^{-2}\left(e^{\kappa\tau}-1\right)^{-2}e^{\kappa\tau} = -\tfrac{1}{2}\sigma^2 e^{\kappa\tau}h^2(\tau). \tag{102A.3}$$

So we have

$$\frac{1}{2}\sigma^2 r H_{rr} - \kappa r H_r - H_\tau = \exp[-rh(\tau)]rh^2(\tau)\left[\frac{1}{2}\sigma^2 + \frac{\kappa}{h(\tau)} - \frac{1}{2}\sigma^2 e^{\kappa\tau}\right]$$
$$= \exp[-rh(\tau)]rh^2(\tau)\left[\frac{1}{2}\sigma^2 + \tfrac{1}{2}\sigma^2(e^{\kappa\tau}-1) - \tfrac{1}{2}\sigma^2 e^{\kappa\tau}\right] = 0 \tag{102A.4}$$

as required establishing the first part of Equation (102A.1).

The state price for the state $r_T = 0$ has the same value as a contract that pays \$1 the first time that the interest rate hits zero because the interest rate is then trapped at zero, and the state $r_T = 0$ will be realized for sure, and with a zero interest rate there is no further discounting. The value of this therefore asset satisfies the usual pricing partial differential equation

$$0 = \tfrac{1}{2}\sigma^2 r Q_{rr} - \kappa r Q_r - rQ - Q_\tau \qquad \text{subject to} \quad Q(0,\tau) = 1 \quad \text{and} \quad Q(r,0) = 0. \tag{102A.5}$$

The boundary conditions are confirmed since $\coth(0) = \infty$. Again differentiating and substituting into Equation (102A.5) gives[27]

[27] The derivative of the hyperbolic cotangent is the negative hyperbolic cosecant function

$$d\coth x/dx = -\text{csch}^2 x = -4(e^x - e^{-x})^{-2}.$$

The third equality in Equation (102A.6) uses the identity $\coth^2 x - \text{csch}^2 x \equiv 1$. The fourth equality follows from the definition of γ.

$$\begin{aligned}
&\frac{1}{2}\sigma^2 r Q_{rr} - \kappa r Q_r - rQ - Q_\tau \\
&\quad = \frac{rQ}{\sigma^2}\left(\tfrac{1}{2}[\kappa - \gamma\coth(\tfrac{1}{2}\gamma\tau)]^2 - \kappa[\kappa - \gamma\coth(\tfrac{1}{2}\gamma\tau)] - \sigma^2 - \tfrac{1}{2}\gamma^2\mathrm{csch}^2(\tfrac{1}{2}\gamma\tau)\right) \\
&\quad = \frac{rQ}{\sigma^2}\left(-\frac{1}{2}\kappa^2 - \sigma^2 + \tfrac{1}{2}\gamma^2[\coth^2(\tfrac{1}{2}\gamma\tau) - \mathrm{csch}^2(\tfrac{1}{2}\gamma\tau)]\right) = \frac{rQ}{\sigma^2}\left(-\tfrac{1}{2}\kappa^2 - \sigma^2 + \tfrac{1}{2}\gamma^2\right) = 0
\end{aligned} \tag{102A.6}$$

as required establishing the second part of Equation (102A.1).

102A.2 Bond Price When $r_t = 0$ Is Accessible for Only the Risk-Neutral Process

As discussed in the body of the paper, the pricing equation is the standard one with only an altered boundary condition setting the bond price to zero when the interest rate is zero. This boundary condition is required to make the risk-neutral and true processes equivalent.

$$0 = \tfrac{1}{2}\sigma^2 r P_{rr} - \kappa r P_r - rP - P_\tau \qquad P(r,0) = 1 \quad P(0,\tau) = 0. \tag{102A.7}$$

The price of the bond is

$$\begin{aligned}
P(r,\tau) &= \exp[-rB(\tau)]\,(1 - \exp[-r\xi(\tau)]) \\
\textit{where} \quad \xi(\tau) &\equiv \frac{2\gamma^2}{\sigma^2[\gamma\sinh(\gamma\tau) + \kappa\cosh(\gamma\tau) - \kappa]} \\
B(\tau) &\equiv \frac{2(1 - e^{-\gamma\tau})}{2\gamma + (\kappa - \gamma)(1 - e^{-\gamma\tau})} = \frac{2}{\gamma\coth(\frac{1}{2}\gamma\tau) + \kappa}.
\end{aligned} \tag{102A.8}$$

$B(\tau)$ is the same function as found in the CIR solution.

The maturity condition are satisfied since $B(0) = 0$, $\xi(0) = \infty$, and the boundary condition at $r = 0$ is clearly satisfied. That the solution satisfies the pricing partial differential equation can be verified by substituting the derivatives

$$\begin{aligned}
P_r &= -B(\tau)P(r,\tau) + \xi(\tau)\exp(-r[B(\tau) + \xi(\tau)]) \\
P_{rr} &= B^2(\tau)P(r,\tau) - \xi(\tau)[\xi(\tau) + B(\tau)]\exp(-r[B(\tau) + \xi(\tau)]) \\
P_\tau &= -rB'(\tau)P(r,\tau) + r\xi'(\tau)\exp(-r[B(\tau) + \xi(\tau)])
\end{aligned} \tag{102A.9}$$

into the partial differential equation and collecting terms

$$0 = rP(r,\tau)\left[\tfrac{1}{2}\sigma^2 B^2 + \kappa B - 1 + B'\right] - r\exp(-r[B(\tau) + \xi(\tau)])\left[\tfrac{1}{2}\sigma^2\xi^2 + (\kappa + \tfrac{1}{2}\sigma^2 B)\xi + \xi'\right]. \tag{102A.10}$$

The first term is zero because $B(\tau)$ is the same function as in the CIR model. So we need only verify that the final term in brackets is also zero. The derivative of ξ is

$$\begin{aligned}\xi'(\tau) &= -\frac{2\gamma^2}{\sigma^2}[\gamma\sinh(\gamma\tau)+\kappa\cosh(\gamma\tau)-\kappa]^{-2}[\gamma^2\cosh(\gamma\tau)+\gamma\kappa\sinh(\gamma\tau)]\\ &= -\frac{\sigma^2}{2\gamma^2}\xi^2(\tau)[\gamma^2\cosh(\gamma\tau)+\gamma\kappa\sinh(\gamma\tau)].\end{aligned}\tag{102A.11}$$

So the final term in brackets in Equation (102A.10) is

$$\begin{aligned}\frac{1}{2}\sigma^2\xi^2+(\kappa+\frac{1}{2}\sigma^2B)\xi+\xi' &= \frac{1}{2}\sigma^2\xi^2\left[1+(2\kappa/\sigma^2+B)/\xi-\cosh(\gamma\tau)-(\kappa/\gamma)\sin(\gamma\tau)\right]\\ &= \frac{1}{2}\sigma^2\xi^2\left[1+\gamma^{-2}\left(\kappa+\frac{1}{2}\sigma^2B\right)[\gamma\sinh(\gamma\tau)+\kappa\cosh(\gamma\tau)-\kappa]-\cosh(\gamma\tau)-(\kappa/\gamma)\sin(\gamma\tau)\right]\\ &= \frac{1}{2}\sigma^2\xi^2\left[\frac{1}{2}\sigma^2\gamma^{-2}B[\gamma\sinh(\gamma\tau)+\kappa\cosh(\gamma\tau)-\kappa]+(\kappa^2/\gamma^2-1)[\cosh(\gamma\tau)-1]\right]\\ &= \frac{1}{2}\sigma^4\xi^2\gamma^{-2}\left[\frac{1}{2}B[\gamma\sinh(\gamma\tau)+\kappa\cosh(\gamma\tau)-\kappa]-\cosh(\gamma\tau)+1\right].\end{aligned}\tag{102A.12}$$

Substituting for $B(\tau)$ and using the "half-angle" identity,[28]

$$\begin{aligned}\frac{1}{2}\sigma^2\xi^2+(\kappa+\frac{1}{2}\sigma^2B)\xi+\xi' &= \frac{1}{2}\sigma^4\xi^2\gamma^{-2}\left[\frac{\gamma\sinh(\gamma\tau)+\kappa\cosh(\gamma\tau)-\kappa}{\gamma[\cosh(\gamma\tau)+1]/\sinh(\gamma\tau)+\kappa}-\cosh(\gamma\tau)+1\right]\\ &= \frac{1}{2}\sigma^4\xi^2\gamma^{-2}\left[\frac{\gamma\sinh^2(\gamma\tau)-\gamma\cosh^2(\gamma\tau)+\gamma}{\gamma[\cosh(\gamma\tau)+1]+\kappa\sinh(\gamma\tau)}\right]=0.\end{aligned}\tag{102A.13}$$

102A.3 Properties of the Affine Exponentially Smoothed Average Model

The multivariate affine model with a exponentially smoothed average has a Kolmogorov backward equation of

$$0=\tfrac{1}{2}\sigma^2 rF_{rr}+\kappa(x-r)F_r+\delta(r-x)F_x-F_\tau.\tag{102A.14}$$

The joint probability distribution for r and x, and other probably functions are the solutions to this partial differential equation subject to various boundary conditions. In particular, the expected values $\mathbb{E}[r_{t+\tau}]$ and $\mathbb{E}\left[r^2_{t+\tau}\right]$ are solutions with boundary conditions $F(r,x,0) = r$ and r^2, respectively. The expected value of the interest rate is

$$\mathbb{E}[r_{t+\tau}]=\frac{r_t[\delta+\kappa e^{-(\kappa+\delta)\tau}]+\kappa x_t[1-e^{-(\kappa+\delta)\tau}]}{\kappa+\delta}\tag{102A.15}$$

The expected value of the square of the interest rate has a quite messy formula, but for times far in the future, its asymptotic behavior is

$$\mathbb{E}[r^2_{t+\tau}]=\frac{\sigma^2\delta^2}{2\kappa(\kappa+\delta)^3}(\delta r_t+\kappa x_t)\tau+o(\tau).\tag{102A.16}$$

28 The "half-angle" identity is $\coth\frac{1}{2}x = (\cosh x+1)/\sinh x$. The final equality in Equation (102A.13) follows from the identity $\cosh^2 x-\sinh^2 x=1$.

Since the mean value converges to $\mathbb{E}[r_\infty] = (\delta r_t + \kappa x_t)/(\delta + \kappa)$, the variance of $r_{t+\tau}$ also diverges at the rate τ.

102A.4 Properties of the Three-Halves Power Interest Rate Process

The steady-state distribution for a diffusion on $(0, \infty)$ with inaccessible boundaries and evolution $dx = \mu(x)dt + \sigma(x)d\omega$ is

$$f(x_\infty) = \frac{C}{\sigma^2(x)} \exp\left[2\int^x \mu(z)/\sigma^2(z)dz\right] \tag{102A.17}$$

where C is the constant of integration required to ensure the density integrates to one. For the three-halves power process the steady state density is

$$\begin{aligned} f(r_\infty) &= \frac{C}{\sigma^2 r^3} \exp\left[2\int^r \kappa z(\theta - z)/\sigma^2 z^3 dz\right] \\ &= (2\kappa\theta/\sigma^2)^{2+2\kappa/\sigma^2} r^{-3-2\kappa/\sigma^2} \exp(-2\kappa\theta/\sigma^2 r). \end{aligned} \tag{102A.18}$$

For the risk-neutral dynamics

$$\begin{aligned} dr &= r[\kappa\theta - \psi_1 - (\kappa + \psi_2)r]dt + \sigma r^{\frac{3}{2}} d\omega \\ &= \hat{\kappa} r(\hat{\theta} - r)dt + \sigma r^{\frac{3}{2}} d\omega, \end{aligned} \tag{102A.19}$$

the bond price is given in Cox et al. (1985) as

$$\begin{aligned} P(r,\tau) &= \frac{\Gamma(\nu - \delta)}{\Gamma(\nu)} [c(\tau)/r]^\delta M(\delta, \nu, -c(\tau)/r) \\ \text{where} \quad c(\tau) &\equiv \frac{2\hat{\kappa}\hat{\theta}}{\sigma^2}\left[e^{(\kappa\theta - \psi_1)\tau} - 1\right]^{-1}, \qquad \hat{\beta} \equiv 1 + 2\hat{\kappa}/\sigma^2 \\ \delta &\equiv \tfrac{1}{2}(\hat{\beta}^2 + 8/\sigma^2)^{1/2} - \tfrac{1}{2}\hat{\beta}, \qquad \nu \equiv 1 + (\hat{\beta}^2 + 8/\sigma^2)^{1/2}, \end{aligned} \tag{102A.20}$$

To help analyze the three-halves power model, we first present some preliminary results. Define the function $m(z; a, b) \equiv z^{-a} M(a, b, -1/z)$ where $M(a, b, x)$ is the confluent hypergeometric function.[29] Then

$$\begin{aligned} m'(z) &= -az^{-a-1} M(a, b, -1/z) + z^{-a-2}(a/b) M(a+1, b+1, -1/z) \\ &= -az^{-a-1} M(a+1, b, -1/z). \end{aligned} \tag{102A.21}$$

By repeated application of Equation (102A.21), the n^{th} derivative is

$$\begin{aligned} m^{(n)}(z) &= (-1)^n a(a+1)\cdots(a+n-1) z^{-a-n} M(a+n, b, -1/z) \\ &= (-1)^n \frac{\Gamma(a+n)}{\Gamma(a)} z^{-a-n} M(a+n, b, -1/z) \\ \text{with} \quad m^{(n)}(0) &= (-1)^n \frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b-a-n)}. \end{aligned} \tag{102A.22}$$

Since derivatives of all orders exist, m is analytic and can be expressed as the power series

$$m(z) = \sum_{n=0}^{\infty} z^n m^{(n)}(0)/n! = \sum_{n=0}^{\infty} (-z)^n \frac{\Gamma(a+n)\Gamma(b)}{n!\Gamma(a)\Gamma(b-a-n)}. \tag{102A.23}$$

[29] The derivative of the confluent hypergeometric function is $\partial M(a, b, x)/\partial x = (a/b) M(a+1, b+1, x)$. The asymptotic behavior as $x \to \infty$ is $M(a, b, -x) = \Gamma(b)/\Gamma(b-a) x^{-a}[1 + O(1/x)]$. See Abramowitz and Stegum (1964).

The bond price in Equation (102A.20) therefore can be written as the power series[30]

$$P(r,\tau) = \frac{\Gamma(\nu-\delta)}{\Gamma(\delta)} m\,(c(\tau)/r;\delta,\nu) = \frac{\Gamma(\nu-\delta)}{\Gamma(\delta)} \sum_{n=0}^{\infty} [-r/c(\tau)]^n \frac{\Gamma(\delta+n)}{n!\Gamma(\nu-\delta-n)}$$
$$= 1 - [r/c(\tau)]\delta(\nu-\delta-1) + \tfrac{1}{2}[r/c(\tau)]^2 \delta(\nu-\delta-1)(1+\delta)(\nu-\delta-2) - \cdots$$
$$= 1 - \frac{2r}{\sigma^2 c(\tau)} + \frac{2(1+\hat{\kappa})r^2}{\sigma^4 c^2(\tau)} + \cdots \tag{102A.24}$$

And since $\ell n(1+x) = x - \frac{1}{2}x^2 + \cdots$, the yield to maturity can be expressed as the power series

$$Y(r,\tau) \equiv -\frac{1}{\tau}\ell n\, P(r,\tau) = \frac{2r}{\sigma^2 \tau c(\tau)} - \frac{2(\hat{\kappa}+1)r^2}{\sigma^4 \tau c^2(\tau)} + \frac{2r^2}{\sigma^4 \tau c^2(\tau)} + \cdots$$
$$= \frac{2r}{\sigma^2 \tau c(\tau)} - \frac{2\hat{\kappa} r^2}{\sigma^4 \tau c^2(\tau)} + O([r/c(\tau)]^3). \tag{102A.25}$$

At short maturities

$$Y(r,\tau) = r + \tfrac{1}{2}\hat{\kappa} r(\hat{\theta} - r)\tau + O(\tau^2). \tag{102A.26}$$

[30] Note that $\delta(\nu-\delta-1) = \sigma^{-4}\left\{\left[\left(\hat{\kappa}+\frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\right]^{1/2} + \hat{\kappa} + \frac{1}{2}\sigma^2\right\}\left\{\left[\left(\hat{\kappa}+\frac{1}{2}\sigma^2\right)^2 + 2\sigma^2\right]^{1/2} - \hat{\kappa} - \frac{1}{2}\sigma^2\right\} = 2\sigma^{-2}$ and $\nu - 2\delta - 2 = 2\hat{\kappa}/\sigma^2$