Searching for Rewards

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Loyalty programs are pervasive across numerous markets, offering members rewards based on their past purchases for future benefits. This study explores the dynamics of loyalty programs within a repeated ordered search framework, where consumers sequentially search for the optimal product across multiple firms over two periods. Our findings reveal that firms strategically use price discounts and rewards to influence consumer behaviors. Price discounts discourage further search in the current shopping period, while rewards encourage consumer loyalty by inducing prominence in subsequent visits. As search costs increase, firms tend to offer lower price discounts but higher rewards. This strategy increases industry profit but reduces consumer surplus. Compared with its absence, loyalty programs decrease both industry profit and consumer welfare, leading to a lose-lose outcome. Moreover, we demonstrate that when the market is heterogeneous, high-type firms, with larger networks, offer lower rewards but achieve higher second-period prices and greater consumer loyalty, contrasting with low-type firms that compensate with higher rewards for their smaller networks. This study offers new insights into the strategic use of loyalty programs and their impact on market competition.

Keywords: Loyalty Program, Consumer Search, Repeated Purchase, Customer Relationship Management, Competitive Strategy
1 Introduction

Loyalty programs are increasingly ubiquitous, demonstrating significant growth in memberships and their perceived value among consumers. A 2016 survey by Accenture (Wollan et al., 2017), revealed that over 90% of companies implement some form of loyalty programs, with membership in the U.S. growing at an impressive annual rate of 26.7%. Such rapid growth underscores the perceived value that these programs offer to consumers, a sentiment strongly supported by a 2019 Wirecard study, which revealed that rewards play a crucial role in the purchasing decisions of the vast majority of consumers.\footnote{Source: https://thelbma.com/research/wirecard-consumer-incentives-2019}

The widespread appeal of loyalty programs spans various industries, underscoring their versatility and impact on business success. From the retail sector’s points-for-purchases schemes, as exemplified by Sephora’s Beauty Insider program,\footnote{The Sephora Beauty Insider program offers points for every dollar spent, which can be exchanged for exclusive products and beauty experiences: https://thelbma.com/research/wirecard-consumer-incentives-2019} to the travel incentives of airline frequent flyer programs like Delta’s SkyMiles, and the fitness rewards offered by platforms such as ClassPass, loyalty programs have been effectively tailored to meet the diverse needs of consumers. Fast food chains, including Starbucks, McDonald’s, and Chipotle have also successfully harnessed loyalty programs to foster repeat business, demonstrating the broad applicability and potential of these marketing tools.

Despite their evident popularity and immediate benefits to consumers, the long-term advantages of loyalty programs for firms remain ambiguous (Dowling and Uncles, 1997; Bombaj and Dekimpe, 2020). One classic argument about why loyalty programs could benefit firms in the long run is based on the switch costs (e.g., Caminal and Matutes 1990 and Kim et al. 2001). Loyalty programs can effectively lock in customers by offering rewards for repeated purchases, potentially leading to increased profits over time. However, the effectiveness of such loyalty programs is often questioned in practice, as critics believe that any advantage they provide can be easily copied by competitors. This can lead to a prisoner’s dilemma scenario, where the costs of maintaining loyalty programs outweigh the benefits, making it hard for firms to maintain long-term benefits from these programs (Deighton and Shoemaker, 2000). Such skepticism is bolstered by observations from Accenture (Wollan et al., 2017), which highlight the hidden costs and managerial challenges associated with maintaining loyalty programs. This contrast in views presents the complex challenge companies face when implementing loyalty programs, highlighting an area that still demands extensive research and investigation. In this paper, we revisit this discussion and examine the role of loyalty programs by explicitly considering consumers’ costly search decisions within the framework of repeated ordered search. Our analysis seeks to formalize and present a new rationale behind
The firms’ adoption of loyalty programs: to gain prominence in future consumer searches.

Imagine the scenario of Clair, who visited NYC and stayed at a Hyatt Hotel. Before her visit, she searched several hotels and compared their prices and rewards offered from their reward programs. Then, she chose to stay at Hyatt Hotel and joined their loyalty program for a discount on future stays. Now, as she plans a trip to San Diego, Clair naturally starts her search with Hyatt, hoping to leverage her loyalty benefits. However, if Hyatt can’t accommodate her specific needs, like a room with two queen beds for a family trip, she’s prompted to consider other brands. The narrative shifts as she plans another trip to Reykjavik, Iceland, where the absence of a Hyatt hotel directly leads her to explore alternative accommodations. These scenarios highlight several important points for the role of reward program, and prominence that it can provide for the focal firm.

When consumers choose a product, they weigh both the current price and potential future rewards from loyalty programs. Such programs not only retain current customers by introducing switching costs but also attract new ones. Fast food chains like McDonald’s and Chipotle, for instance, use their loyalty apps as a strategy to draw new patrons. Furthermore, loyalty programs elevate a brand’s prominence in a consumer’s future searches, exemplified by Hyatt becoming the first place to search for someone like Clair planning her next trip. Consumers begin their search with the prominent brands, moving on to other options only if the initial choice fails to meet their expectations or value criteria.

The effect of loyalty program is amplified by the brand’s network size; the larger it is, the more enticing the loyalty program, encouraging customers to start their search with the brand. While Hyatt’s loyalty program initially steers Clair to consider Hyatt for her accommodations, its absence in Reykjavik for her Iceland trip starkly illustrates the constraints of a limited network. This scenario underscores the critical importance of network size; Hyatt’s inability to serve Clair in Iceland due to a lack of local presence directly impacts her loyalty and search behavior. The larger the network, the more attractive the loyalty program becomes, highlighting the strategic expansion of brand networks to maximize loyalty program benefits. Indeed, many brands operate all over the world, and a single loyalty program applies to all of them. The more extensive the brand chain is, the more appealing its loyalty program becomes to consumers. Clair will find Hyatt’s loyalty program more attractive, when she can redeem rewards at more travel destinations.

Such scale economy in loyalty program is vividly illustrated by Marriott’s acquisition of Starwood Preferred Guest (SPG) and McDonald’s ambitious goal to expand to 50,000 locations worldwide by 2027. These examples reflect the broader principle that network size

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3Ad Age, “How McDonald’s and other restaurant brands are driving loyalty apps in face of inflation,” May 24, 2023: https://adage.com/article/marketing-news-strategy/how-mcdonalds-chipotle-and-others-are-promoting-loyalty-apps/2496401

4“McDonald’s eyes speedy ramp-up to 50,000 restaurants by 2027” https://hospitality.economictimes.
significantly impacts the appeal and effectiveness of loyalty programs, a concept we explore through our economic model to understand the nuances of loyalty in the hospitality industry.

By analyzing the interplay between loyalty programs, consumer search behavior, and the size of brand networks, our paper seeks to illuminate the strategic considerations firms must weigh in designing these programs. Specifically, we investigate how firms use loyalty programs to compete for future prominence, identifying the critical differences between pricing and rewards. We further examine how these programs influence competitive dynamics, pricing strategies, future rewards, and the overall welfare of the market.

To answer these questions, we adopt a sequential searching framework developed by Wolinsky (1986), where consumers discover a specific match value for each firm at a search cost sequentially. We extend this framework to a two-period repeated ordered search model where consumer preferences are independent across time. In this model, cities are filled with firms, each potentially having a branch, defined by an activation rate that captures the firm’s network size or branch presence across cities. Consumers, located randomly in these cities in each period, engage with firms to learn about prices, rewards, and the scope of brand networks, necessitating discovery in both periods due to potential variations in product offerings.

We first characterize the pure-strategy symmetric equilibrium when all firms have the same network size, exploring how competition is affected by the search cost and the activation rate. Our analysis reveals that higher search costs and larger networks both encourage repeat purchases, yet their impacts on competition differ. As search costs increase, consumer’s incentive to search diminishes, leading them to repeat purchase from the same firms, thereby placing higher value on loyalty rewards. This prompts firms to intensify competition by enhancing reward offerings. Interestingly, the relationship between search costs and the first period pricing is complex; prices may rise with moderate search costs but could drop when costs are high to lock in consumers. In the second period, the combination of greater obstacles to switching and enhanced rewards means firms can charge higher prices. This scenario results in higher industry profits but a decrease in consumer surplus due to a shift towards purchases motivated more by rewards than by preference.

Then, we analyze how activation rates affect market competition. A surprising finding is that reward levels remain constant regardless of these rates. The efficacy of loyalty programs is consistent across all cities a firm operates, and the size of a firm’s network doesn’t alter the likelihood of a member switching to a competitor within a city. This stability means there’s no incentive for firms to adjust their reward design. Overall, a higher activation rate leads more members being locked in, so firms compete more intensively in the first period and
exploit them with higher prices in the second period. The industry profit drops because the market becomes more competitive. The total surplus decreases due to lower match inefficiency associated with the locked-in effect of rewards. The consumer surplus only increases when firms compete very intensively—that is when the search cost and the activation rate are high; otherwise, it decreases as driven by match efficiency loss.

To further distinguish the roles of the price and the committed reward, we numerically analyze the equilibrium when the market is heterogeneous, that is firms have different network sizes. While both price and reward can influence the number of customers a firm can attract, rewards specifically play a crucial role in customer retention, affecting the likelihood of repeat purchases. The first period price, on the other hand, seeks to optimize a firm’s immediate and future profits through demand. Firms with extensive networks (high-type firms) tend to offer smaller rewards, as their wide reach across numerous cities naturally attracts new members. These firms rely less on rewards for customer retention, given their advantage in network size. Conversely, firms with smaller networks (low-type firms) use larger rewards, addressing their competitive shortfall. The strategy around the first-period pricing varies significantly with market conditions. In scenarios where attracting members yields long-term profitability—especially at high search costs—firms with larger networks may set lower prices to draw in customers early. However, these high-type firms typically charge more in the second period, leveraging their larger base of loyalty members to secure higher revenues.

The rest of the paper is organized as follows: We begin with a literature review in the next section, followed by a detailed description of the model in Section 3. The main results, including equilibrium analysis for both homogeneous and heterogeneous network sizes among firms, are discussed in Section 4 and in Section 5. We conclude with a summary of our findings and discussion in Section 6. All the technical proofs are provided in the Appendix.

2 Literature Review

Our research contributes to several strands of the literature, starting with area of customer relationship management (CRM) and loyalty program research. Loyalty programs is widely used and one of the most important tools in CRM to enhance customer retention and profitability (Deighton and Shoemaker, 2000; Belli et al., 2022). Several papers investigate the design of optimal reward program, with a particular emphasis on the referral reward (Biyalogorsky et al., 2001; Wolters et al., 2020; Kamada and Öry, 2020; Fourie et al., 2023), or focusing on the role of switching costs created from reward program as suggested by Caminal and Matutes (1990) and Kim et al. (2001). Our model also builds on the switching costs (see Farrell and Klemperer (2007) for a comprehensive review). Von Weizsäcker (1984) and Klemperer (1987) investigate competition with exogenous switch costs. In contrast, rewards
from loyalty programs can be regarded as a way to endogenize such switch costs (Caminal and Matutes, 1990), and we also consider how loyalty programs introduce these costs endogenously.

Caminal and Matutes (1990) analyze a duopoly setting with consumers randomly change preference across two periods. Firms can precommit a discount for customers re-purchase in the second period. Then, switch cost endogenously arises in equilibrium, and profits increase with reward commitment. Beyond switching costs, loyalty programs are analyzed as commitment devices (Kim et al., 2001, 2004). Kim et al. (2001) argue that firms choose reward types as a means of committing future price. When there are large segments of price-insensitive consumers, firms are profitable to provide inefficient reward to price high in the future. Furthermore, Kim et al. (2004) shows that in scenarios of limited capacity and low demand, firms can use rewards to moderate competition, reducing the incentive to undercut prices. These programs, by offering rewards for repeated purchases, not only aim to lock in customers but also present a strategic tool for firms to soften competition by committing reward.

Our study further connects with CRM research on how firms promote to loyal customers, and the impact of loyalty on price competition. Notably, Narasimhan (1988) was among the first to explore how price promotions differ between loyal and non-loyal customer segments. Also, Villas-Boas (2004) demonstrated that an increase in loyal customers could paradoxically lower future firm profitability due to customer uncertainty and value distribution. Building on these insights, Kuksov and Zia (2021) recently examined how firms in a duopoly set search costs for non-loyal customers. We extend these literature by considering search friction under competition. We find that rewards from loyalty programs, serving as endogenous switching costs, intensify with exogenous search costs. This exploration of how external search frictions and internal reward strategies influence firm profits in a competitive context marks a novel addition to the literature.

Our model builds on the consumer sequential search theory in product market. Wolinsky (1986) was the first to propose a sequential search in a horizontally differentiated market in the context of random search order. Anderson and Renault (1999) analyzed how search costs moderate the relationship between the degree of product differentiation and equilibrium prices. A more recent line of research has examined more realistic market situations where search is non-random and consumers search in a deliberate order (see Armstrong (2017) for an extensive literature on this ordered search literature). Armstrong et al. (2009) demonstrates that, when consumers engage in costly search across firms, prominence, or being the prominent.
first shopping destination, can be valuable as it can preempt demand. In this paper, authors analyze a situation where for each consumer, only one firm is prominent and the others are symmetrically non-prominent. Thus, the consumer searches the prominent one first and, if it is not satisfactory, she will continue to search randomly among other non-prominent firms. Unlike classic one-period ordered search models, which require the equilibrium to be asymmetric so that consumers follow a non-random search order. Here, the initial enrollment in loyalty programs creates a natural asymmetry among firms in subsequent periods, effectively integrating the concept of ordered search over multiple market interactions.

Subsequently, several papers investigate on how firms can become prominent for consumer search. Firms use various instruments to direct consumer searches; for example, by charging lower price (Chen and He, 2011; Armstrong and Zhou, 2011), advertisement (Haan and Moraga-González, 2011; Mayzlin and Shin, 2011), brand positioning (Ke et al., 2023), targeting (Shin and Yu, 2021), offering service (Shin, 2007; Janssen and Ke, 2020), and providing price or product information (Choi et al., 2018; Lu, 2023). The work most closely related to ours is by Armstrong and Zhou (2011), which considers several ways for a firm to gain prominence, notably through securing an initial sale to leverage the default bias, where consumers tend to revert to previously chosen firms. This concept of default bias shares parallels with our model’s consumer search behaviors. However, unlike Armstrong and Zhou (2011), who examines consumers search in a deterministic order and prominence is the same for all consumers, our model considers prominence as a reward-driven, endogenous outcome. Here, each firm becomes prominent only for its members in subsequent periods. We further show how firms seek to compete for prominence under different levels of search friction.

3 A Model of Repeated Ordered Search

We examine a market characterized by monopolistic competition with $N \to \infty$ firms (or brands). There are $B \in \mathbb{Z}^+$ cities, where each firm has an active branch in $\gamma B$ cities, where the activation rate $\gamma \in (0, 1]$ captures the breadth of each firm’s network size. The presence of a firm’s branch in specific cities is independent across firms. For each firm $j \in \{1, \cdots, N\}$, we denote $B(j) \subset \{1, \cdots, B\}$ as the realized set of cities where it actively maintains branches. In these locations, the firm offers one product, with the marginal production cost normalized to be zero.

There are $M$ consumers. We consider a two-period model, where a representative consumer travels randomly to city $b(t) \in \{1, \cdots, B\}$ in period $t = 1, 2$. The travel destinations are independent across consumers. The consumer’s match value with firm $j$’s product at branch $b(t)$ is denoted by $v^j_t$, which follows a uniform distribution in $[0, 1]$ and is indepen-
dently distributed across both $j$ and $t$.\(^6\)

At first glance, it seems a strong assumption to have the two products offered by the same firm in two branches to be independent, because in reality they may share some common attributes. It’s important to note that such common attributes known before searching—like a hotel chain’s reputation—are accounted for in our model. For example, two hotels by Hyatt may share similar reputation in design features, but it is possible that consumers recognize this before search and their search focus on the idiosyncratic features of each hotel such as locations or restaurant options. Moreover, we assume that even when consumers make repeated purchases from the same branch, the perceived value can vary across periods, necessitating further search. For instance, while the value offered by the Hyatt Hotel in NYC might be consistently high, the specific appeal of the hotel can change based on factors unique to each visit, such as the convenience of its location relative to event venues. A consumer might have chosen the Hyatt previously because it hosted a conference they attended, but its proximity to future events of interest remains uncertain until searched for again. Similarly, a customer’s needs can change between visits. On one occasion, she might visit Starbucks for coffee, and on another, for a muffin. Given that Starbucks’ offerings can be perceived as varying between visits, the customer must undertake a new search to determine the current value of the products according to her changing needs.

Without loss of generality, we can normalize the average number of consumers per city per firm, $M/(N \cdot B)$ to one. The model also incorporates a time discounting factor $\delta \in (0, 1]$. Figure 1 illustrates the market structure with an example. Each house icon represents an active branch. In this example, firm 1 has a branch in city 1 and 3; firm 2 has a branch in city 1, 2 and B, and so on.

Each firm, from 1 to $N$, sets a consistent price $p^j_t$ for its products across all its branches for each period $t = 1, 2$.\(^7\) This approach mirrors practices of major brands like McDonald’s or fitness chains, which may adjust prices over time but maintain price consistency across different locations. Moreover, firms offer a reward $r^j \geq 0$ to consumers who purchase in the first period, valid for use in any branch during the second period. This setup includes the possibility of a firm opting out of a loyalty program by setting $r^j = 0$, effectively making the

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\(^6\)Notice that intuitively, $v^j_t$ depends on $b(t)$; however, to simplify notation, this dependence relationship is not signified explicitly, but rather, through the superscript of $t$.

\(^7\)We require each firm to set the same price for all its branches in each period. This assumption is less restrictive than it appears for three reasons. First, given all branches of the firm are symmetric, the prices for all branches could be the same in equilibrium, even if we allow the firm to set different prices for different branches. Second, since each consumer visits only one branch per period, differing prices across branches wouldn’t be noticeable to them. Essentially, the single-price per period assumption stipulates consumers’ off-equilibrium belief about the prices of all other branches of the same firm to be the same as the branch they visited. Last, the real-world pricing strategies of widespread brands, which aim for uniformity across regions to maintain brand consistency and customer fairness, provide a practical foundation for our model’s assumption.
In each period, consumers purchase at most one product from a firm. Consumers’ utility from purchasing firm \( j \)’s product in city \( b(t) \) during period \( t \) is:

\[
    u_t^j = \begin{cases} 
    v_t^j - p_t^j + r^j, & \text{if } t = 2 \text{ and enrolled in firm } j \text{'s loyalty program,} \\
    v_t^j - p_t^j, & \text{otherwise.}
    \end{cases}
\]

Consumers have an outside option of not buying in each period, which is normalized to be zero. Before deciding on a purchase, consumers need to search among firms to discover prices and match values, as this information is \textit{a priori} unknown. Thus, in each period, they conduct a sequential search among firms with perfect recalls before making a purchase decision (Wolinsky 1986). The search cost per firm is assumed to be \( s > 0 \).

In the first period, when a consumer visits firm \( j \) in city \( b(1) \), she discovers her match value \( v_1^j \), the price \( p_1^j \), and reward \( r^j \). Based on this information, she decides whether to buy and enroll in the loyalty program. There is zero cost on the consumer side for enrolling, which implies that consumers will always enroll as long as \( r^j > 0 \). In the second period, by visiting firm \( k \) in city \( b(2) \), the consumer observes her match value \( v_2^k \) as well as the price \( p_2^k \), and she decides whether to make a purchase. If she previously bought from the same firm \( (k = j) \), she can redeem the reward of \( r^j \) if \( k = j \). The connected lines with arrows in Figure 1 showcase one sample trajectory of consumer search. The consumer travels to city 1 in the first period by searching firm 1 first and then firm N before purchasing at firm 2; she travels to city 2 in the second period by searching firm 2 first and then firm 3 and N.

\[\text{Figure 1: Illustration of Market Structure and a Sample Consumer Search Trajectory.}\]
3.1 Equilibrium Concept

In our analysis, we use the perfect Bayesian equilibrium (PBE) as our solution concept. Since all firms are ex-ante the same, it is natural to focus on the symmetric equilibrium with \( p_j^t = p_i^t \) and \( r_j = r^* \) for \( t = 1, 2 \) and \( \forall j \in \{1, \cdots , N\} \). In this equilibrium, consumers search randomly in the first period, but in the second period, prominence emerges endogenously due to reward programs, leading consumers to first search the firm from which they have a reward.

Consistent with existing literature (Wolinsky, 1986; Anderson and Renault, 1999), we assume that consumers hold passive beliefs such that if they observe a firm’s deviation from the equilibrium behavior, they continue to believe that the other firms stick to their equilibrium paths. Moreover, unique to our repeated ordered search setting, we also need to specify consumers’ off-equilibrium belief on a firm’s second-period price when they observe the firm’s deviation on its first-period price and/or reward. There are two cases to consider.

The first case concerns with a consumer who made a purchase from the deviating firm in the first period. The refinement by sequential equilibria (Kreps and Wilson, 1982) requires that the consumer updates her belief of the second-period price \( p_j^2 \) based on the firm’s deviation in the first period, \( p_j^1 \) and/or \( r_j \). The second case concerns with a consumer who did not purchase from the deviating firm but instead, decided to continue to search. We specify the off-equilibrium belief with the following assumption.

**Assumption 1.** If a consumer visited firm \( j \) in the first period but did not purchase from it after observing that it deviated on \( p_j^1 \) and/or \( r_j \), we assume that, prior to any visits in the second period, the consumer still believes that firm \( j \) will adhere to the second-period equilibrium price \( p_j^2 \).

This assumption is based on a few key points. First, with infinite number of firms in the market, it’s realistic to expect that consumers might not remember every detail given limited attention or memory, especially about firms they only considered but didn’t buy from. For instance, Clair, planning her next trip to San Diego, might not recall the specific offers from New York hotels she browsed but didn’t book. Second, one can show that in absence of Assumption 1, there does not exist a pure-strategy symmetric equilibrium. If all other firms set the equilibrium price and rewards \( (p_i^1 \) and \( r^*) \), an individual firm will have incentives to deviate \( p_i^1 \) infinitesimally, which would lead to an infinitesimal change of its profit in the current period but a demand jump in the next period, as all consumers who have browsed but did not purchased from this firm will expect it to charge a slightly lower price, \( p_j^2 < p_i^2 \). Thus, they would prefer to visit this deviating firm only after the firm they have loyal membership with but before all other firms. Lastly, a more profound point is that Assumption 1 eliminates the possibility that firms could use their first-period pricing to shape consumer expectations about second-period prices, thereby guiding consumer search.
in the second period. In other words, Assumption 1 isolates rewards as the only instrument in equilibrium to direct consumer search for their next purchases, reflecting our thinking that loyalty programs are offered to direct consumer search for their future purchases.

4 Equilibrium Analysis

To begin our analysis, we define the reservation value $w$ by equating the search cost $s$ with the option value from searching (Weitzman 1979):

$$s = \int_{w}^{1} (v - w) dv = \frac{1}{2} (1 - w)^2 \Rightarrow w = 1 - \sqrt{2} s \in (0, 1) \text{ for } s \in (0, 1/2).$$

By definition, the reservation value $w$ represents a consumer’s threshold for being indifferent between accepting a guaranteed offer immediately or continuing to search among firms for a better deal.

Next, we present the following lemma.

**Lemma 1.** If a pure-strategy symmetric perfect Bayesian equilibrium exists, we must have that $0 < r^* \leq w$.

Lemma 1 implies that in equilibrium, all firms will offer loyalty programs, and thus, consumers search randomly in the first period but in subsequent searches, they search the firm first where they have memberships.

We will solve the game by backward induction below. Particularly, we analyze the situation an individual firm $j$ unilaterally deviates to $p^j_1$, $p^j_2$ and $r^j$. As all of its branches behave in the same way, it suffices to consider one branch’s optimization. We will identify the condition that guarantees that the firm has no incentive to deviate. This confirms the existence of the symmetric equilibrium and also pins down the equilibrium.

4.1 Second Period

In the second period, from firm $j$’s perspective, consumers are categorized into two groups: (1) **loyal customers**, who made purchases and joined the loyalty program of firm $j$ in the first period, and (2) **guest visitors**, who neither bought from nor joined the loyalty program of firm $j$. We will analyze the demand firm $j$ receives from each type of consumer separately.

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9If firm $j$ deviate from the equilibrium, other firms may respond optimally afterward, potentially deviating from equilibrium price $p^j_2$. However, as there are infinite firms in each city, firm $j$’s deviation does not affect demand for other firms. Therefore, they will adhere to equilibrium strategies and consumers hold rational expectation on that.
Demand from Loyal Customers

For firm $j$, the total demand in the first period across all its branches is denoted by $\gamma BD^j_1$, where $D^j_1$ is the consumer demand for one branch of firm $j$ in period 1. Consequently, in the second period, a portion $\gamma D^j_1$ of members will travel to a specific city where one of branches is located.

Consider a consumer who is a loyal customer of firm $j$. She already observed $p^j_1$ and $r^j$ in the first period, based on which, she forms an expectation of the firm’s second-period price, $\tilde{p}^j_2$. The consumer will search firm $j$ first if and only if

$$w - \tilde{p}^j_2 + r^j \geq w - p^*_2 \iff \tilde{p}^j_2 \leq p^*_2 + r^j.$$  

We will verify the above condition after determining $\tilde{p}^j_2$ below. Intuitively, the consumer understands that firm $j$ offers the reward to attract loyal customers to visit it first in the second period, so it is unprofitable for the firm to set the second-period price so high such that no consumer visits. After the consumer visits firm $j$, she learns $v^j_2$ and will purchase if and only if $v^j_2 - p^*_2 \geq w - p^*_2$. Otherwise, the consumer will continue to search and never come back. Therefore, we have the loyal customers’ demand for one branch of firm $j$ in the second period as

$$D^j_{2L} = \gamma D^j_1 \left[ 1 - \left( w - p^*_2 + p^j_2 - r^j \right) \right].$$

Demand from Guest Visitors

Consider a guest visitor of firm $j$. If the firm she previously purchased from is present in the market (with probability $\gamma$), she first visits that firm $k$, where she holds a membership, and learns $v^k_2$. If the value $v^k_2 - p^*_2 + r^*$ is less than $w - p^*_2$ (with probability $\Pr(v^k_2 - p^*_2 + r^* < w - p^*_2) = w - r^*$), she decides to continue searching. Given her continuation, with probability $1/(\gamma N - 1) \cdot w^n$, she visits $n$ other firms before visiting firm $j$. She will purchase from firm $j$ if and only if $v^j_2 - p^*_2 \geq w - p^*_2$, which occurs with probability $1 - (w - p^*_2 + p^j_2)$.

On the other hand, if the firm where she has membership isn’t active (with probability $1 - \gamma$), she starts searching randomly among active firms. With probability $1/(\gamma N) \cdot w^n$, she visits $n$ other firms before visiting firm $j$ and will purchase from firm $j$ if and only if $v^j_2 - p^*_2 \geq w - p^*_2$, which also occurs with probability $1 - (w - p^*_2 + p^j_2)$. In summary, the demand from guest visitors for firm $j$ in the second period is

$$D^j_{2G} = \lim_{N \to \infty} \left( \frac{M}{B} - \gamma D^j_1 \right) \left( \gamma (w - r^*) \sum_{n=0}^{\gamma N - 2} \frac{1}{\gamma N - 1} w^n + (1 - \gamma) \sum_{n=0}^{\gamma N - 1} \frac{1}{\gamma N} w^n \right) \times \left[ 1 - \left( w - p^*_2 + p^j_2 \right) \right].$$
Maximizing this profit leads to the optimal second-period price:

\[
\pi_2^j(p_2^*) = \frac{(\gamma(w - r^*) + 1 - \gamma) \left(1 - \left(w - p_2^* + p_2^j\right)\right)}{\gamma(1 - w)}.
\]

**Optimal Second-Period Price**

To find the optimal price for the second period, we analyze the profit of firm \(j\), which includes demands from both loyal customers and guest visitors. Firm \(j\)’s second period profit function is

\[
\pi_2^j(p_2^*) = D_{2L}^j(p_2^* - r^j) + D_{2G}^j p_2^j
\]

By substituting \(p_2^* = D_{2L}^j(p_2^* - r^j)\), we can obtain the firm’s optimal second-period profit, \(\pi_2^j(D_1^j, r^j) \equiv \pi_2^j(p_2^*(D_1^j, r^j))\). By taking derivatives, one can easily prove the following lemma that illustrates the dependence relationship of \(\pi_2^j(D_1^j, r^j)\) and \(\pi_2^j(p_2^*(D_1^j, r^j))\) on \(D_1^j\) and \(r^j\).

**Lemma 2.** The optimal second-period price \(p_2^j(D_1^j, r^j)\) increases with both demand \(D_1^j\) and reward \(r^j\); the second-period profit \(\pi_2^j(D_1^j, r^j)\) increases with demand \(D_1^j\) and decreases with reward \(r^j\).

The lemma suggests that while a higher first-period demand consistently benefits the firm by boosting both price and profit, an increase in the reward, though positively affecting the price, negatively impacts the profit in the second period. To get an intuition behind the optimal second-period price, \(p_2^j(D_1^j, r^j)\), we can rewrite equation (1) as the following:

\[
p_2^j(D_1^j, r^j) = \lambda(D_1^j) \left(\frac{1 - w + p_2^*}{2} + r^j\right) + (1 - \lambda(D_1^j)) \frac{1 - w + p_2^*}{2},
\]

where \(\lambda(D_1^j) \equiv D_{2L}^j/[D_{2L}^j + (1 - \gamma + \gamma(w - r^*))/(\gamma^2(1 - w))]\). Note that the optimal prices for loyal customers and guest visitors alone are \((1 - w + p_2^*)/2 + r^j\) and \((1 - w + p_2^*)/2\), respectively. Therefore, \(p_2^j(D_1^j, r^j)\) can be seen as a linear combination of these two. The weight on price for loyal customers \(\lambda(D_1^j)\) is the fraction of loyal customers among who come to visit firm \(j\).
The part of the price for loyal customers, \((1 - w + p^*_2)/2 + r^j\), is higher because they receive a reward, \(r^j\). As the first period demand, \(D^j_1\), increases, indicating more loyal customers, the optimal price shifts closer to what is ideal for loyal customers, hence increasing. An increase in the reward, \(r^j\), directly raises the price component for loyal customers, thereby lifting the overall optimal price, \(p^*_j(D^j_1, r^j)\).

Regarding profit, \(\pi^*_j(D^j_1, r^j)\), an increase in loyal customer number, \(D^j_1\), allows the firm to raise prices, leading to a higher profit in the second period. This incentivizes firms to compete by lowering their first-period prices to acquire more consumers in the first period, which we will formally analyze next. The reward \(r^j\) affects profit through two channels. First, with a higher reward, loyal customers are more likely to stop searching and make a purchase, leading to a higher demand. On the other hand, the firm would need to spend more on rewarding loyal customers. Overall, the latter effect dominates and the optimal second-period profit decreases as the reward increases.

4.2 First Period

In the first period, consumers search randomly. After visiting firm \(j\) and observing the value \(v^j_1\), price \(p^j_1\) and reward \(r^j\), a consumer can infer the first period demand \(D^j_1\). Based on this information, a consumer forms an expectation of the firm’s second-period price, \(\tilde{p}^j_2 = p^*_2(D^j_1, r^j)\) given by equation (1). Then, the consumer stops searching and makes a purchase from firm \(j\) if and only if

\[
v^j_1 - p^j_1 + \delta E[u^j_2|j] \geq w - p^*_1 + \delta E[u^j_2|k],
\]

where \(k\) represents an alternative firm that consumer may buy from if she decides to continue to search; \(E[u^j_2|j]\) and \(E[u^j_2|k]\) are the consumer’s expected utility in the second period if she purchases from firm \(j\) and \(k\) in the first period, respectively. This equation (2) implies that the consumer’s optimal search strategy in the first period still follows an index policy, as the expected utility \(E[u^j_2|j]\) and \(E[u^j_2|k]\) are independent of the consumer’s first-period match value realization.

By definition, we have,

\[
E[u^j_2|j] = \gamma \left[ -s + \int_{w-p^*_2+p^*_j(D^j_1,r^j)-r^j}^{1} \left( v^j_2 - p^*_2 \right) \, dv^j_2 \right] + \left( w - p^*_2 \right),
\]

\[
E[u^j_2|k] = \gamma \left[ -s + \int_{w-r^*}^{1} \left( v^k_2 - p^*_2 + r^* \right) \, dv^k_2 + \left( w - r^* \right) \left( w - p^*_2 \right) \right] + \left( 1 - \gamma \right) \left( w - p^*_2 \right).
\]
By substituting the above expressions of \( E[u_2|j] \) and \( E[u_2|k] \) back to equation (2), we have that a consumer stops at firm \( j \) if and only if \( v_j \geq \hat{v}(p_1^j, r^j) \), where

\[
\hat{v}(p_1^j, r^j) \equiv w - p_1^* + p_1^j - \delta (E[u_2|j] - E[u_2|k])
\]

\[= w - p_1^* + p_1^j - \frac{\gamma \delta}{2} \left( p_2^* - p_2^j(D_1^j, r^j) + r^j - r^* \right)
\]

\[\times \left( p_2^* - p_2^j(D_1^j, r^j) + r^j + r^* + 2 - 2w \right).
\]

(3)

Similar to the case for guest visitors’ demand in the second period, we can calculate the first-period demand as:

\[D_1^j = \frac{1 - \hat{v}(p_1^j, r^j)}{\gamma(1 - w)}.\]

(4)

Equation (4) indicates that \( D_1^j \) depends on \( \hat{v}(p_1^j, r^j) \), which is affected by \( p_2^j(D_1^j, r^j) \). Since \( p_2^j(D_1^j, r^j) \) itself also depends on \( D_1^j \), by combining equations (1), (3) and (4), one could obtain \( D_1^j \) as a function of \( p_1^j \) and \( r^j \). However, it turns out the resulting expression of \( D_1^j \) is very complicated. On the other hand, by combining equations (3) and (4), one can express \( p_1^j \) as a more manageable function of \( D_1^j \) and \( r^j \) explicitly:

\[p_1^j = p_1^* + (1 - w) \left( 1 - \gamma D_1^j \right) + \frac{\gamma \delta}{2} \left( p_2^* - p_2^j(D_1^j, r^j) + r^j - r^* \right)
\]

\[\times \left( p_2^* - p_2^j(D_1^j, r^j) + r^j + r^* + 2 - 2w \right) \equiv p_1^j(D_1^j, r^j).
\]

Thus, it is more straightforward to consider firm \( j \)'s decisions in terms of \( D_1^j \) and \( r^j \) rather than \( p_1^j \) and \( r^j \). In other words, one could equivalently view firm \( j \) making decisions on \( D_1^j \) and \( r^j \) in the first period.

Firm \( j \)'s profit in the first period, \( \pi_1^j(D_1^j, r^j) = D_1^j p_1^j(D_1^j, r^j) \). In the first period, the firm tries to maximize its total profit \( \pi_1^j(D_1^j, r^*) = \pi_1^j(D_1^j, r^j) + \delta \pi_2^j(D_1^j, r^j) \), where

\[(D_1^j, r^*) = \arg \max_{(D_1^j,r^j)} \pi_1^j(D_1^j, r^j).
\]

We demonstrate that when the search cost is not prohibitively high, there exists a unique symmetric equilibrium where rewards arise endogenously.

**Proposition 1.** A unique symmetric equilibrium exists when \( 0 < s < \pi \equiv 1/[2(2 + \gamma)^2] \). In this equilibrium, the firms’ equilibrium reward and prices are

\[r^* = \sqrt{2s},\]

\[p_1^* = \frac{\sqrt{2s} - 4\gamma s + 2\sqrt{2(1 - 2\delta)}\gamma^2 s^\frac{3}{2}}{(1 - \gamma \sqrt{2s})^2},\]

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Moreover, in this equilibrium, consumers randomly search in the first period, make purchases, and become members. In the second period, when the firms they have memberships with are active in the cities they travel to, they first visit them and continue searching randomly if necessary. If the firms are not active, they randomly search among all firms.

The proposition indicates that firms endogenously adopt loyalty programs to compete for future prominence. Further, we examine how these equilibrium outcomes change in response to the key market conditions in the following two propositions. The first proposition shows the impact of search costs on various market outcomes.

**Proposition 2.** The impact of search cost, \( s \), on first-period price and profit, \( p^*_1 \) and \( \pi^*_1 \), varies depending on the discount factor \( \delta \), and the activation rate \( \gamma \). Specifically,

1. when \( 0 < \delta \leq 1/2 \), or \( 1/2 < \delta < 1 \) and \( 0 < \gamma \leq \hat{\gamma} \), both \( p^*_1 \) and \( \pi^*_1 \) increases with \( s \);

2. when \( 1/2 < \delta < 1 \) and \( \hat{\gamma} < \gamma < 1 \), \( p^*_1 \) and \( \pi^*_1 \) first increases and then decreases with \( s \), where \( \hat{\gamma} \) is uniquely determined by the equation of \( \delta \gamma^3 + 3\delta \gamma^2 - 2 = 0 \).

Moreover, other equilibrium outcomes including \( r^* \), \( p^*_2 \), second-period profit \( \pi^*_2 \), total profit \( \pi^*_T \) and industry profit \( \pi^*_I \equiv \gamma NB\pi^*_T \) all increase with \( s \), while consumer surplus \( CS \) and total surplus \( TS \) both decrease with \( s \).

Figure 2 illustrates Case 2 in Proposition 2. As the search cost increases, consumers are less inclined to explore multiple options. Given that loyal customers first search firms with which they have memberships (when active), the likelihood of repeat purchases rises. Therefore, firms are more incentivized to attract consumers with higher rewards. In this context, the search cost acts as an exogenous switching cost for loyal customers, whereas rewards represent an endogenous switching cost determined by firms. Our findings reveal that endogenous switching costs effectively complement exogenous ones. Rewards, pre-committed at the outset, serves as another channel firms compete with each other beyond pricing. With elevated search costs, firms compete more fiercely through loyalty programs.

The impact of search friction on the first-period price exhibits a non-linear pattern for large values of \( \delta \) and \( \gamma \). Specifically, as search friction becomes higher, the first-period price, \( p^*_1 \), initially rises before declining, as illustrated by Figure 2-(a). The first-period price directly affects first-period profits and indirectly influence second-period profits through demand of the first period. Conversely, reducing prices to attract more demand secure a larger member base for the subsequent period, reinforced by increased rewards, \( r^* \). At lower search costs, where consumer search is more intense, reducing prices to secure loyalty is less compelling.
Instead, adjusting prices upward in response to increased friction enhances first-period profits, leading to greater overall profitability. On the other hand, when search costs are relatively high, locking in consumers through lower price is more effective as they are more very likely to stop searching and purchase there, resulting in a price curve that initially rises then falls with increasing $s$. However, when either $\delta$ or $\gamma$ is relatively low, firms prioritize immediate profits, leading to a consistent rise in the first-period price as search costs increase.

For the second period pricing, as search costs increase, consumers’ propensity to explore decreases, allowing firms to charge higher prices. This observation aligns with the finding from Wolinsky (1986), where higher search friction reduces market competition and leads to higher prices. Moreover, as rewards rise in response to greater search friction, loyal customers are more inclined to repurchase from the same firms, enabling these firms to further increase their prices. Consequently, the second-period price rises with search costs, as illustrated by Figure 2-(a).

The first period profit mirrors the response of $p_1^*$ to changes in search cost. The second-period profit, the total profit and industry profit increase as the search cost being higher. This uptrend is attributed to consumers’ diminished willingness to engage in extensive search efforts, effectively augmenting each firm’s market power. As a result, the total surplus diminishes alongside with increasing search frictions. Initially, this reduction in surplus is due to the increased costs faced by consumers. Subsequently, repeat purchases increases driven more by higher rewards, not by consumer preference, leading to a higher incidence of mismatches. With more mismatches occur, the total surplus decreases. A similar rationale applies to the decline in consumer surplus. These patterns are illustrated in Figure 2-(b) and (c), respectively.

Next, we investigate the impact of activation rate (which captures the network size) on the equilibrium outcomes.

**Proposition 3.** An increase in the activation rate, $\gamma$, leads to a decrease in $p_1^*$, while $r^*$
remain unchanged, and $p_2^*$ increases. Profits for the first period and second period $\pi_1^*$, $\pi_2^*$, total profit $\pi_T^*$, industry profit $\pi_I^*$, and total surplus $TS$ all decrease with $\gamma$. Consumer surplus $CS$ reacts differently depending on $s$; it decreases when $0 < s \leq s_1$ or $s_2 \leq s < 1/8$, but shows a decrease followed by an increase when $s_1 < s < s_2$, where the threshold $s_1$ and $s_2$ are uniquely determined by the equation $1 - 3\sqrt{2}s - 18s + 6\sqrt{2}s^2 = 0$ and $\sqrt{2} - 6\sqrt{5} + 12s^2 = 0$ respectively, both of which fall within $[0, 1/8]$. $s_1 \approx 0.022$, $s_2 \approx 0.078$.

The proposition shows that, counterintuitively, the reward remains constant regardless of the activation rate. Essentially, rewards serve as a price discount to prevent loyal customers from seeking alternatives in each city, functioning effectively at the local level regardless of a firm’s network size. The prevalence or scarcity of a firm’s branches across cities does not influence the effectiveness of rewards; they operate independently at each branch location. With an increase in the activation rate, consumers in the second period are more likely to remain with the firms where they hold memberships, resulting in a decline in the number of guest visitors. This allows firms to charge a higher price to exploit loyal customers. However, the incentive to acquire more customers in the first period heightens competition, leading to a reduction in first-period prices as the activation rate increases. See Figure 3-(a) for the impact of activation rate on equilibrium rewards and prices.

![Figure 3: Comparative Statics Examples on $\gamma$ When $\delta = 1$ and $s = 0.057$.](image)

Furthermore, the activation rate directly affects the number of active firms within each city. A higher activation rate translates to more firms per city. With a constant flow of consumers to each city, each firm’s market share shrinks. Therefore, the first-period profit, second-period profit, and the total profit all decrease. The industry profit also decreases, because firms compete too fiercely for customer acquisition (see Figure 3-(b) for profit results). The total surplus diminishes with the activation rate for reasons akin to its decrease with search costs: more repeat purchases are motivated by rewards rather than genuine product preference, leading to welfare losses due to valuation mismatches. The impact on consumer surplus is nuanced and contingent on the search cost magnitude. At moderate search costs,
two opposing effects emerge. Initially, increased valuation loss from a higher activation rate suppresses consumer surplus. However, as the rate climbs further, reduced prices in the first period benefit consumers, thereby enhancing consumer surplus. On the other hand, at very low search costs, extensive consumer search in the second period and the likelihood of brand switching prompt firms to be very conservative in pricing low in the first period, leading to the loss of consumer surplus. Conversely, at very high search costs, increasing mismatch decreases the consumer surplus as in Proposition 2. Overall, consumer surplus strictly decreases with the activation rate. Figure 3-(c) illustrates the impact of activation rate on the firm profits, total surplus and consumer surplus.

Discussion: The Effects of Search Cost and Activation Rate

It might appear that higher search costs and increased activation rates impose similar influence on market competition by promoting repeated purchases and reducing the propensity for consumers to switch brands in the second period. Thus, fewer switching behaviors in the second period increases each firm’s market power but intensifies competition in the first period as member acquisition becomes more profitable in the long run. However, there are notable distinctions in their impact on competition.

First, while rewards increases with search costs, they remain unaffected by changes in the activation rate. Although both factors lead to more repeat purchases, heightened search costs discourage consumers from exploring alternatives within a city, whereas increased activation rates primarily makes directed searches occur more frequently across all different cities. Rewards, thus, play a pivotal role in each city where a firm operates, with firms opting to increase rewards only when there’s a high likelihood of members ceasing their search to make repeat purchases (for example, when search costs become higher). This does not occur with variations in the activation rate, resulting in rewards remaining constant. Second, higher activation rates makes the market more competitive by reducing firm’s market shares, whereas increased search costs, by increasing each firm’s market power even in the first period, mitigate the competitive incentive for aggressively acquiring new customers.

These analyses underscore the distinct impacts of search cost and activation rate on market equilibrium, as depicted in the corresponding Figures 2 and 3, highlighting how these factors shape market competition and equilibrium outcomes differently.

4.3 Impact of Loyalty Programs

To investigate the impact of loyalty programs, we consider a benchmark scenario of without loyalty program, where firms only decide their prices in each period. Also, the match value of products are independent across time in the absence of loyalty program. This setting renders
prior consumer behaviors irrelevant to future decisions, allowing the game to be simplified into two separate stage games.

Lemma 3. Without loyalty program, there is a unique symmetric equilibrium where the prices are $p_1^B = p_2^B = \sqrt{2}s$.

Then, we further compare the outcomes with and without loyalty programs, as summarized in the following proposition, where the superscript “B” denotes the benchmark case.

Proposition 4. Comparison of equilibria with and without loyalty programs:

- $p_1^* < p_1^B$, $p_2^* > p_2^B$ and $p_2^* - r^* < p_2^B$.
- $\pi_1^* < \pi_1^B$, $\pi_2^* > \pi_2^B$, $\pi_T^* < \pi_T^B$, $\pi_I^* < \pi_I^B$, $CS < CS^B$ and $TS < TS^B$.

This analysis reveals that loyalty programs introduce an additional dimension of competition for firms. By offering rewards, firms effectively reduce consumer propensity to switch in the second period, encouraging more aggressive pricing strategies in the first period to secure consumer loyalty.

In the absence of loyalty programs, consumer randomly searches in the second period. However, with loyalty programs in place, consumers prioritize firms where they hold memberships. Firms, accordingly, differentiate their pricing: offering a reduced rate, $p_2^* - r^*$, for members, and a standard rate, $p_2^*$, for non-members. Here, $p_2^*$ is always higher than the price of without loyalty program ($p_2^* > p_2^B$) because firms want to exploit loyal customers who visit them first. Nonetheless, the effective price for members, after accounting for rewards, is lower than $p_2^B$ (i.e., $(p_2^* - r^*) < p_2^B$), indicating that firms engage in competition by offering rewards to members. This allows firms to leverage their prominence among members to impose higher prices and secure greater profits than that of without loyalty programs.

In the first period, with loyalty programs, firms have incentive to reduce their prices to attract members, anticipating future exploitation. This contrasts with scenarios without loyalty programs, where consumer searches randomly in the second period. Consequently, both the first period’s price and profit are lower than those from the benchmark case of without loyalty programs ($p_1^* < p_1^B$ and $\pi_1^* < \pi_1^B$). Loyalty programs thus spur competition over prominence, not solely through rewards but also via the first period price. This intensified price competition results in diminished the total and industry profits ($\pi_T^* < \pi_T^B$, $\pi_I^* < \pi_I^B$).

Without loyalty programs, consumer purchases are based purely on high product valuation following random searches. In contrast, loyalty programs encourage members to prioritize affiliated firms, sometimes leading to purchases even when product fit is suboptimal. This discrepancy suggests that rewards may deter consumers from discovering products that better match their preferences, resulting in welfare losses and consumers are also worse off when
loyalty programs are adopted. To summarize, loyalty programs lead to a lose-lose situation for both consumers and firms.

5 Heterogeneous Network Sizes

In this section, we extend the basic model to allow for two distinct activation rates of firms in the market, denoted by $\theta \in \{H,L\}$. Specifically, $\alpha \in (0,1)$ fraction of firms are of high type with $\theta = H$ who operate a branch in $\gamma_H B$ cities, where a high activation rate $\gamma_H \in (0,1]$ symbolizes an extensive network coverage. The remaining $1 - \alpha$ fraction of firms are of low type with $\theta = L$ who operate a branch in $\gamma_L B$ cities, where a low activation rate $\gamma_L \in (0, \gamma_H]$ represents more limited network sizes. Again, whether a firm operates a branch in specific cities is independent across firms. Consumers do not observe the firms’ types a priori. Upon visiting firm $j$ in city $b(1)$ in the first period, consumers observe the firm’s type $\theta_j$, along with which, her match value $v_{j1}$, the price $p_{j1}^{\theta_j}$, and reward $r_{j1}^{\theta_j}$.

In a similar spirit of Assumption 1, we introduce an additional assumption to address consumers’ potential memory or attention limitations regarding firms they’ve visited without making a purchase in this setting:

**Assumption 2.** If a consumer visited a firm in the first period without purchasing, in the second period, they no longer remember the firm’s type $\theta_j$.

This assumption effectively homogenize all firms from which they did not purchase, significantly streamlining the consumer decision-making process for subsequent searches. Specifically, it suggests that if consumers opt to search beyond the firm with whom she has reward in the second period, they will search randomly among all other firms. Absent this assumption, the decision of which firms to visit next — be it previously visited high-type firms, low-type firms, or those of unknown type—becomes notably more complex.

5.1 Equilibrium Analysis

Similar to the main model, we start solving the game backward. Our analysis focuses on an equilibrium where all firms of the same type $\theta$, set the same rewards $r_{\theta}^*$ and price $p_{1\theta}^*$ for both periods. Similar to main analysis, we characterize the equilibrium by considering potential deviations by any firm of type $\theta$ to alternative reward and pricing and strategies $(r_{j}^{\theta}, p_{1\theta}^{j}, p_{2\theta}^{j})$.

5.1.1 Second Period Analysis

Again, there are two kinds of consumers in the second period: (1) Loyal customers and (2) Guest Visitors. We first calculate firm $j$’s demand from these two types of consumers
separately and then, derive the optimal second period prices for them.

**Demand from Loyal Customers**

Consider the total number of active firms in the second period as \( N_A = (\alpha \gamma_H + (1 - \alpha) \gamma_L) N \). The proportions of high-type and low-type firms active in a city are defined respectively as:

\[
\beta_H = \frac{\alpha \gamma_H}{\alpha \gamma_H + (1 - \alpha) \gamma_L} = \frac{\alpha \gamma_H}{A_2}, \quad \beta_L = \frac{(1 - \alpha) \gamma_L}{\alpha \gamma_H + (1 - \alpha) \gamma_L} = \frac{(1 - \alpha) \gamma_L}{A_2},
\]

where \( A_2 \equiv \alpha \gamma_H + (1 - \alpha) \gamma_L \) is the *composite activation rate* in the second period, which quantifies the overall market presence of both high and low-type firms in the second period, taking into account their respective shares in the market and their individual probabilities of operating an active branch in any given city. This measure simplifies our expressions, thereby facilitating a more streamlined analysis of heterogeneous network size case.

Define \( w_2 \) as the second-period reservation value, the utility level at which a consumer is indifferent between receiving a sure payoff \( w_2 \) and continuing searching randomly from firms she does not have membership. We obtain this reservation value by equating the marginal cost of searching, \( s \), with the marginal benefit of searching one additional firm:

\[
s = \beta_H \int_{p_2^H + w_2}^{1} (v_{j2}^2 - p_2^*H - w_2) dv_{j2} + \beta_L \int_{p_2^L + w_2}^{1} (v_{j2}^2 - p_2^*L - w_2) dv_{j2},
\]

which yield

\[
w_2 = 1 - \frac{\alpha \gamma_H}{A_2} p_2^*H - \frac{(1 - \alpha) \gamma_L}{A_2} p_2^*L - \sqrt{2s - \frac{\alpha (1 - \alpha) \gamma_H \gamma_L (p_2^*H - p_2^*L)^2}{A_2^2}}.
\]

Similar to the main model, for firm \( j \), the total demand in the first period across all its branches is captured by \( \gamma_\theta D_{1\theta}^j B \). This translates to \( \gamma_\theta D_{1\theta}^j \) members destined for a particular city where the firm operates a branch. Consider a loyal customer who holds a membership with firm \( j \). She observes \( p_{1\theta}^j \) and \( r_{\theta}^j \) in the first period, based on which, she forms an expectation of the firm’s second-period price, \( \tilde{p}_{2\theta}^j \). The consumer will search firm \( j \) first if and only if \( w - \tilde{p}_{2\theta}^j + r_{\theta}^j \geq w_2 \). Conditional on searching there, a purchase occurs strictly when the utility from buying, \( v_{j2}^2 - p_{2\theta}^j + r_{\theta}^j \), exceeds the reservation utility \( w_2 \), leading to the following expression for the second period demand of type \( \theta \) from loyal customers:

\[
D_{2\theta L}^j(p_{2\theta}^j) = \gamma_\theta D_{1\theta}^j \left[ 1 - \left( w_2 + p_{2\theta}^j - r_{\theta}^j \right) \right]. \tag{5}
\]

**Demand from Guest Visitors**

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In contrast to the main model, guest visitors now fall into four distinct segments. These segments are determined based on their memberships with either high-type (H) or low-type (L) firms and the operational status (active or not) of the firms they are affiliated with.

For a guest visitor who is a member of an active high-type firm, she first visits her affiliated firm. If her net utility \( w_2^i - p_{2H}^i + r_H^* \) is less than the reservation value \( w_2 \), she proceeds to search other firms. Given her continuation, with probability \( 1/(N_A-1) \cdot (\beta_H(w_2 + p_{2H}^i) + \beta_L(w_2 + p_{2L}^i))^n \), she visits \( n \) other firms before visiting firm \( j \); she will purchase from firm \( j \) if and only if \( w_j^i - p_{2g}^j \geq w_2 \). This leads to a specific demand formula for high-type firms with active memberships. A similar decision-making pathway applies to guest visitors of active low-type (L) firms, resulting in a parallel set of demand equations as follows:

\[
D_{2gH,A}^i = \lim_{N \to \infty} D_{1L}^* \gamma_H (1-\alpha) \gamma_H N (w_2 + p_{2H}^i - r_H^*) \\
\times \sum_{n=0}^{N_A-2} \frac{(\beta_H(w_2 + p_{2H}^i) + \beta_L(w_2 + p_{2L}^i))^n}{N_A-1} \left( 1 - w_2 - p_{2g}^j \right) \\
= D_{1L}^* \gamma_H \beta_H (w_2 + p_{2H}^i - r_H^*) \left( 1 - w_2 - p_{2g}^j \right),
\]

\[
D_{2gL,A}^i = \lim_{N \to \infty} D_{1L}^* \gamma_L (1-\alpha) \gamma_L N (w_2 + p_{2L}^i - r_L^*) \\
\times \sum_{n=0}^{N_A-2} \frac{(\beta_H(w_2 + p_{2H}^i) + \beta_L(w_2 + p_{2L}^i))^n}{N_A-1} \left( 1 - w_2 - p_{2g}^j \right) \\
= D_{1L}^* \gamma_L \beta_L (w_2 + p_{2L}^i - r_L^*) \left( 1 - w_2 - p_{2g}^j \right),
\]

where \( D_{2gH} \) and \( D_{2gL} \) represent the demand of type \( \theta \) from guest visitors in the second period, as indicated by the prefix "2G". The subscripts HA and LA identify the firm type \( \theta \in \{H, L\} \), and the "A" signifies that the consumer's affiliated firm from the first period is "active" in the current city.

For guest visitors associated with high-type (H) firms that are not active in the current city, they visit \( n \) other firms before visiting firm \( j \) with probability \( 1/N_A \cdot (\beta_H(w_2 + p_{2H}^i) + \beta_L(w_2 + p_{2L}^i))^n \), and purchase from firm \( j \) if and only if the utility from doing so, exceeds a reservation value, \( w_2 \). \( w_j^i - p_{2g}^j \geq w_2 \). Similarly, guest visitors linked to low-type (L) firms, where the firm is also not active, exhibit a parallel decision-making process. These lead to the demand \( D_{2gH} \) and \( D_{2gL} \) for non-active high-type and low-type, respectively.

\[
D_{2gH}^i = \lim_{N \to \infty} D_{1H}^* \gamma_H (1-\alpha) \gamma_H N \sum_{n=0}^{N_A-1} \frac{(\beta_H(w_2 + p_{2H}^i) + \beta_L(w_2 + p_{2L}^i))^n}{N_A} \left( 1 - w_2 - p_{2g}^j \right) \\
= \frac{D_{1H}^* \gamma_H (1-\alpha) \gamma_H}{A_2 (1 - (\beta_H(w_2 + p_{2H}^i) + \beta_L(w_2 + p_{2L}^i)))} \left( 1 - w_2 - p_{2g}^j \right),
\]
The optimal second-period price can be obtained by:

\[ D_{2G, GN} = \lim_{N \to \infty} D^*_1, \gamma_L(1-\alpha)(1-\gamma_L)N \sum_{n=0}^{N-1} \left( \frac{\beta_H(w_2 + p_{2H}^j) + \beta_L(w_2 + p_{2L}^j)}{N} \right)^n (1-w_2-p_{29}^j) \]

\[ = \frac{D^*_1, \gamma_L(1-\alpha)(1-\gamma_L)}{A_2(1-(\beta_H(w_2 + p_{2H}^j) + \beta_L(w_2 + p_{2L}^j)))} (1-w_2-p_{29}^j). \]

By adding these demands from all four guest visitor segments: active high-type \((D_{2G, HA}^j)\), active low-type \((D_{2G, LA}^j)\), non-active high-type \((D_{2G, HN}^j)\), and non-active low-type \((D_{2G, LN}^j)\), we can calculate the total demand from guest visitors \(D_{2G}^j\) faced by firm \(j\) of type \(\theta\). This total demand reflects the combined effects of firm type and operational status in a heterogeneous market environment.

\[ D_{2G}^j = D_{2G, HA}^j + D_{2G, LA}^j + D_{2G, HN}^j + D_{2G, LN}^j \]

\[ = \frac{(1-w_2-p_{29}^j)}{A_2(1-(\beta_H(w_2 + p_{2H}^j) + \beta_L(w_2 + p_{2L}^j)))} \left( D^*_1, \alpha \gamma_H^2(w_2 + p_{2H}^j - r_H^j) + D^*_1, (1-\alpha) \gamma_L^2(w_2 + p_{2L}^j - r_L^j) \right) \]

\[ + \frac{D^*_1, \alpha \gamma_H (1-\gamma_H) + D^*_1, (1-\alpha) \gamma_L (1-\gamma_L)}{A_2(1-(\beta_H(w_2 + p_{2H}^j) + \beta_L(w_2 + p_{2L}^j)))}. \]  

(6)

**Optimal Second-Period Price**

To find the optimal price for the second period, we analyze the profit of firm \(j\) of type \(\theta\), which includes demands from both loyal customers and guest visitors from equations (5) and (6). Firm \(j\)’s second period profit function is \(\pi_{29}^j(p_{29}^j) = \pi_{29}^j(p_{29}^j)D_{2G, LN}^j + p_{29}^j D_{2G, GN}^j\). The firm’s optimal second period price can be obtained by

\[ p_{29}^j(D_{1\theta}, r_{\theta}^j) = \arg \max_{p_{29}^j} \pi_{29}^j(p_{29}^j) \]

\[ = \frac{1-w_2}{2} + \frac{\gamma_H(1-w_2-p_{2H}^j) + (1-\alpha)\gamma_L(1-w_2-p_{2L}^j)}{A_2(1-(\beta_H(w_2 + p_{2H}^j) + \beta_L(w_2 + p_{2L}^j)))} D_{1H}^j r_H^j \]

\[ + \frac{\alpha(D_{1H}^j - D_{11H}^j)\gamma_H^2(1-w_2-p_{2H}^j) + \alpha \gamma_H D_{1H}^j(1-\gamma_H^j r_H^j)}{A_2(1-(\beta_H(w_2 + p_{2H}^j) + \beta_L(w_2 + p_{2L}^j)))}. \]

By substituting \(p_{29}^j(D_{1\theta}, r_{\theta}^j)\) back to \(\pi_{29}^j(p_{29}^j)\), we can obtain the firm’s optimal second-period profit, \(\pi_{29}^j(D_{1\theta}, r_{\theta}^j) \equiv \pi_{29}^j(p_{29}^j(D_{1\theta}, r_{\theta}^j))\).

**5.1.2 First Period Analysis**

In the first period, a consumer randomly searches from all firms. Upon visiting a firm \(j\) and learning about the match value \(v_j^i\), first-period price \(p_{1\theta}^i\), and reward offered \(r_{\theta}^i\), consumers form expectations regarding the firm’s pricing and reward strategy for the second period,
denoted by \( \hat{p}_{2\theta}^j = p_{2\theta}^j(D_{1\theta}^j, r_{\theta}^j) \). We define the first period reservation value as \( w_1 \). Formally,

\[
s = \beta_H \int_{p_{1H}^* - \delta E[u_{2H}|H] + w_1}^{1} \left( v_1^j - p_{1H}^* + \delta E[u_{2H}|H] - w_1 \right) dv_1^j + \beta_L \int_{p_{1L}^* - \delta E[u_{2L}|L] + w_1}^{1} \left( v_1^j - p_{1L}^* + \delta E[u_{2L}|L] - w_1 \right) dv_1^j,
\]

where \( E[u_{2H}|H] \) and \( E[u_{2L}|L] \) are the consumer’s expected utility in the second period depending on whether they purchased from a high-type or a low-type firm in the first period, respectively. Where \( E[u_{2\theta}|j] \) is the consumer’s expected utility in the second period conditioning on that she has purchased from firm \( j \) of type \( \theta \) in the first period. We have

\[
E[u_{2\theta}|j] = \gamma_\theta \left[ -s + \int_{w_2 + p_{2\theta}^j(D_{1\theta}^j, r_{\theta}^j)}^{1} \left( v_2^j - p_{2\theta}^j(D_{1\theta}^j, r_{\theta}^j) + r_{\theta}^j \right) dv_2^j + (w_2 + p_{2\theta}^j(D_{1\theta}^j, r_{\theta}^j) - r_{\theta}^j) \right] w_2 + (1 - \gamma_\theta)w_2.
\]

Therefore,

\[
w_1 = 1 - \frac{\alpha \gamma_H \Psi_{1H} + (1 - \alpha) \gamma_L \Psi_{1L}}{A_2} - \sqrt{\frac{A_2}{2s} - \frac{\alpha(1 - \alpha) \gamma_H \gamma_L (\Psi_{1H} - \Psi_{1L})^2}{A_2^2}},
\]

where \( \Psi_{1H} \equiv (p_{1H}^* - \delta E[u_{2H}|H]) \) and \( \Psi_{1L} \equiv (p_{1L}^* - \delta E[u_{2L}|L]) \).

The consumer stops at firm \( j \) if and only if \( v_j \geq \hat{v}_\theta(p_{1\theta}^j, r_{\theta}^j) \equiv w_1 + p_{1\theta}^j - \delta E[u_{2\theta}|j] \). Therefore, the first-period demand is

\[
D_{1\theta}^j = \frac{1 - \hat{v}_\theta(p_{1\theta}^j, r_{\theta}^j)}{(1 - \beta_H(w_1 + \Psi_{1H}) - \beta_L(w_1 + \Psi_{1L})) A_2}.
\]

The price \( p_{1\theta}^j \) can then be written as a function of the first-period demand \( D_{1\theta}^j \) and the reward \( r_{\theta}^j \), providing a direct link between pricing, demand, and the reward structure:

\[
p_{1\theta}^j(D_{1\theta}^j, r_{\theta}^j) \equiv 1 - w_1 + \delta E[u_{2\theta}|j] - A_2(1 - \beta_H(w_1 + \Psi_{1H}) - \beta_L(w_1 + \Psi_{1L})) D_{1\theta}^j.
\]

Firm \( j \)'s profit in the first period is then represented as \( \pi_{1\theta}^j(D_{1\theta}^j, r_{\theta}^j) = D_{1\theta}^j p_{1\theta}^j(D_{1\theta}^j, r_{\theta}^j) \). The optimal strategy for firm \( j \) in the first period involves maximizing the total profit \( \pi_{1\theta}(D_{1\theta}^j, r_{\theta}^j) \), which includes both the first and second-period profits. Thus, the firm’s decision
problem in the first period is,

\[ (D^*_1, r^*_1) = \arg \max_{(D^*_1, r^*_1)} \pi_{1\theta}(D^*_1, r^*_1) = \pi_{1\theta}(D^*_1, r^*_1) + \delta \pi_{2\theta}(D^*_1, r^*_1). \]

\( D^*_1 \) and \( r^*_1 \) should satisfy two optimality conditions for each type respectively. Unfortunately, there is no closed-form solution; instead, we solve the equilibrium numerically, which we present next.

5.1.3 Numerical Analysis: Optimal Rewards and First Period Price

This section outlines our approach to solving the model under a variety of market conditions, providing insights into the practical implications of our theoretical framework. To explore the model’s predictions across different scenarios, we conduct a numerical analysis. We set \( \alpha = 0.5 \) and \( \delta = 1 \), and examine the model outcomes under three specific conditions by varying \( \gamma_H, \gamma_L \), and \( s \):

- With \( s = 0.02 \), \( \gamma_L = 0.1 \), we vary \( \gamma_H \) from 0.1 to 1 in increments of 0.05.
- With \( s = 0.02 \), \( \gamma_H = 1 \), we adjust \( \gamma_L \) from 0.05 to 1 in increments of 0.05.
- Setting \( \gamma_H = 1 \), \( \gamma_L = 0.8 \), we change \( s \) from 0.005 to 0.055 in increments of 0.005.

The results from these analyses, presented in the subsequent figures (see Figure 4), illustrate the robustness of our model across a spectrum of market dynamics.

**Finding 1** (Comparison of High-Type and Low-Type Firms) \( r^*_H < r^*_L \) and \( p^*_2H > p^*_2L \). \( p^*_1H < p^*_1L \) when \( s \) is relatively large, and \( \gamma_H - \gamma_L \) is small; while \( p^*_1H > p^*_1L \) otherwise. \( D^*_1H > D^*_1L \). \( \pi^*_TH > \pi^*_TL \).

The comparison of the high-type and the low-type firms shows that high-type firms (with larger networks) set lower rewards \( r^*_H \) than low-type firms \( r^*_L \), charge higher second-period prices \( p^*_2H \) than \( p^*_2L \), and attract more first-period demand \( D^*_1H \) leading to higher total profits \( \pi^*_TH \) compared to low-type firms. Also, the first-period price \( p^*_1H \) is lower for high-type firms when search costs are significant and the difference in network sizes is minimal, indicating a strategic trade-off between initial price reduction for member acquisition and subsequent reward setting for customer retention.

The numerical analysis reveals intriguing contrasts in optimal reward levels between high-type (larger network) and low-type (smaller network) firms. Specifically, low-type firms set higher rewards (\( r^*_L > r^*_H \)) despite having smaller networks. This is in stark contrast to previous result, which suggested that optimal rewards would not vary with network size under
homogeneous market competition (Proposition 3). Specifically, in a homogeneous setting, all firms, sharing identical network sizes, face equivalent competitive advantage from the network size, leading to uniform reward levels. However, in a heterogeneous scenario, the disparity in network sizes becomes a critical factor that can affect their customer acquisition, leading to different reward levels and prices depending on their network size. High-type firms with extensive networks can afford to offer smaller rewards, as their wide reach across numerous cities naturally attracts new members. Thus, they leverage their extensive network size to acquire new customers efficiently in the first period with lower rewards, allowing them to exploit their loyal customer base with higher prices in the second period ($p_{2H}^* > p_{2L}^*$). On the other hand, low-type firms compensate for their smaller networks with higher rewards to acquire customers in the first period.

While both price reductions and reward increases can attract more customers, they serve distinct roles in a firm’s strategy and they are not necessarily interchangeable completely. In particular, rewards not only attract first-period customers but also enhance second-period retention and profitability, emphasizing the unique value of rewards in sustaining customer loyalty and encouraging repeat business.

Interestingly, first-period price dynamics ($p_{1H}^*$ vs. $p_{1L}^*$) are shaped by the delicate balance between search costs and the differences in network sizes. In situations where securing early membership is crucial for long-term gains, larger network firms may opt for lower initial prices to attract customers promptly. This strategy becomes particularly effective when search costs are high, ensuring member loyalty, and when the network size disparity is minimal, necessitating aggressive competition. Consequently, high-type firms, leveraging their extensive networks, offer more competitive first-period prices than their low-type counterparts. Conversely, when high-type firms enjoy a substantial network advantage or when the significance of loyal customer segments diminishes (due to lower search costs), they might prioritize immediate gains over long-term member acquisition, leading to higher first-period prices. Overall, this dynamic underscores the distinct roles of pricing and rewards in shaping competitive strategies and market outcomes in a heterogeneously networked market.

The numerical analysis’s comparative statics further illustrate how market conditions influence competition between firms with varying network sizes. These findings are summarized below and illustrated by Figure 4.

**Finding 2** *(Comparative statics on $\gamma_H$, $\gamma_L$ and $s$)*

1. As $\gamma_H$ increases, $p_{1H}^*$ increases then decreases, and $p_{1L}^*$ decreases; $r_H^*$ decreases, and $r_L^*$ increases; $p_{2H}^*$ and $p_{2L}^*$ increase; both $\pi_{TH}^*$ and $\pi_{TL}^*$ decrease.
2. As $\gamma_L$ increases, both $p_{1H}^*$ and $p_{1L}^*$ increase then decrease; $r_H^*$ decreases then increases, and $r_L^*$ decreases; $p_{2H}^*$ decreases then increases, and $p_{2L}^*$ increases; both $\pi_{TH}^*$ and $\pi_{TL}^*$. 


3. As \( s \) increases, first-period prices increase then decrease, rewards, second-period prices and total profits increase.

The numeric analysis highlights how changes in network sizes (\( \gamma_H \) and \( \gamma_L \)) and search costs (\( s \)) impact firm strategies and market outcomes. For high-type firms, as their network size (\( \gamma_H \)) grows, they initially increase first-period prices due to their enhanced market presence but eventually decrease prices to intensify member acquisition as competition with other high-type firms becomes their focus. Low-type firms consistently lower their prices in response to high-type firms’ dominance. The rewards strategy adapts accordingly, with high-type firms decreasing rewards while low-type firms increase theirs to maintain competitiveness. Second-period prices for both firm types rise with \( \gamma_H \), reflecting a market shift.
towards homogeneity with predominant high-type firms, which escalates competition and affects overall industry profit and consumer welfare negatively.

As the network size of low-type firms (\(\gamma_L\)) increases, there will be more active low-type firms in the second period, affecting both high and low-type firms’ strategies. Initially, with fewer low-type firms, high-type firms dominate, leading to intense competition among themselves. However, as \(\gamma_L\) rises, the market becomes more diversified, softening competition and prompting all the firms (both high and low types) to adjust their first-period prices and rewards strategically. This differentiation peaks at intermediate \(\gamma_L\) levels, and is more homogenous either \(\gamma_L\) is very low or very high.\(^{10}\) Consequently, industry profits and consumer welfare fluctuate, highlighting a nuanced interplay between market structure and firm strategies. High-type firms adapt by adjusting rewards and prices to maintain their competitive edge and customer base, while low-type firms strive to become more competitive, affecting overall market outcomes.

The comparative statics related to search costs align with findings from homogeneous firms scenarios (as shown in Proposition 2), indicating that first-period prices increase initially but then decrease as search costs rise, while both rewards and second-period price increases with search costs. This pattern suggests that total profit grows while consumer and total surplus decline, mirroring earlier analyses. A notable additional insight that we get here is the differing impacts on high and low-type firms. As search costs increase, the advantage of high-type firms becomes more pronounced, since the likelihood of their future prominence translating into customer purchases is greater in a market with more frictions. Thus, with a higher search cost, high-type firms, with their market prominence, gain more as search costs increase, widening the profitability gap between them and low-type firms. This highlights the strategic importance of network size and search costs in shaping market dynamics and firm profitability.

6 Conclusion

This study presents an economic analysis of loyalty programs, exploring their impact on consumer search behaviors, market competition, and firm profitability across different network sizes. Our findings shed light on the strategic value of loyalty programs as tools for enhancing competitive advantage and building customer loyalty. Theoretically, we examine the role of loyalty programs by explicitly considering consumers’ costly search decisions within the framework of repeated ordered search. Our analysis offers a new rationale behind firms’ adoption of loyalty programs: to gain prominence in future consumer searches. Practically,

\(^{10}\)When there are fewer low-type firms, high-type firms compete with other high-types intensively. So, the market becomes effectively homogeneous dominated by the high types. Conversely, when \(\gamma_L\) approaches \(\gamma_H\), then both types becomes homogeneous.
our research delivers valuable insights for managers seeking to develop more effective loyalty programs. It highlights the importance of achieving a strategic equilibrium between initial pricing incentives and long-term rewards, demonstrating that this balance can significantly influence consumer loyalty and enhance firm profitability.

We first characterize the equilibrium reward program (optimal prices and reward) and examine how the network size and market friction can affect their equilibrium choices and market outcomes under competition. As search costs increases, firms compete more intensively through loyalty programs, leading higher rewards and lower first-period prices to secure more customers initially, allowing the firm to exploit their loyalty in the second period through higher prices. Thus, our analysis reveal that rewards, endogenous switching costs, and exogenous market friction such as search costs are strategic complements. Interestingly, optimal reward levels remain consistent across different network size under homogenous market competition, indicating the uniform effectiveness of loyalty programs regardless of a firm’s operational scale. Nevertheless, an increase in the activation rates leads to a scenario where firms will have more loyal customers in the second period, prompting firms to compete more aggressively initially and leverage this with higher prices to exploit loyal customers. Consequently, industry profits decline due to heightened competition, and while the overall surplus diminishes due to increased valuation losses from mismatches induced by rewards, consumer surplus only sees an increase under conditions of intense competition, i.e., when search costs and activation rates are high.

To further distinguish the roles of the price and the committed reward, we numerically analyze the scenario when the market is heterogeneous, that is firms differ in their network sizes. Our numerical analysis reveals that firms with large networks, or high-type firms, tend to offer smaller rewards, utilizing their broad presence across multiple cities to naturally attract new customers. Conversely, smaller firms, or low-type firms, offset their narrower reach by providing larger rewards to overcome their competitive limitations. This scenario not only highlights the advantage of scale in loyalty program competition but also demonstrates the strategic edge larger firms have in influencing consumer choices without the need for significant rewards. Our examination sheds light on the distinct roles of pricing and rewards in making loyalty programs attractive and effective. While both elements are vital for developing a loyal customer base, the focused use of rewards emerges as a key factor in encouraging ongoing patronage, emphasizing the strategic significance of carefully designed reward mechanisms. The first-period pricing varies greatly according to market conditions. In scenarios where attracting members yields long-term profitability, especially at high search costs, firms with larger networks may set lower prices to draw in customers early. However, these high-type firms typically charge more in the second period, leveraging their larger base of loyalty members to secure higher revenues.
However, our study is not without limitations. First, the intricate nature of our analysis meant we couldn’t analytically explore the variation in firms’ network sizes with a closed-form approach; we resorted to numerical analysis instead. Additionally, our model’s assumptions, such as uniform pricing across all network branches within each period and simplified consumer decision-making in a two-period framework, might not entirely reflect the nuanced realities of actual market behavior. Future studies could enhance our understanding by incorporating varied pricing strategies across different branches, examining the effects of diverse search costs, and considering the benefits of adaptive pricing models over longer periods to boost the effectiveness of loyalty programs further. Future research should also aim to analytically investigate variations in network size and validate our findings.

In conclusion, our research highlights the critical importance of loyalty programs, offering both theoretical and managerial insights into their influence on consumer decisions and market competition. As consumer preferences and market conditions continue to change, the comprehensive impact of loyalty programs will persist as a significant subject of interest for scholars and industry professionals alike. Additionally, the economic and regulatory implications drawn from our study underscore the importance of thoughtfully assessing how loyalty programs affect consumer welfare and overall market health.
Appendix

Proof of Lemma 1

Proof. We first prove $r^* > 0$ by contradiction. Suppose $r^* = 0$, under which, consumers search randomly in both periods. It is profitable for an individual firm $j$ to deviate by setting $r_j > 0$ but very small. This leads to a demand jump in the second period because consumers who purchased from firm $j$ in the first period will start their search at firm $j$ in the second period. Next, we prove $r^* \leq w$ by contradiction. Suppose $r^* > w$, under which, all consumers—even those with the realized match value as zero—will buy at the firm where they have membership with. This implies that an individual firm $j$ has incentives to deviate by setting $r_j = w$, which will not change consumer behaviors but leads to a saving on rewards.

Proof of Proposition 1

Proof. • First, following arguments in the main text, we characterize optimal strategies with necessary conditions. Then, we prove that $r^*, p_1^*$ and $p_2^*$ indeed construct an equilibrium.

We want to characterize symmetric equilibrium. That is, $p_1 = p_1^*$, $p_2 = p_2^*$ and $r = r^*$ in equilibrium. According to equation (4), we can derive that $D_1^* = \frac{1}{\gamma}$. $D_1^*$ and $r^*$ should satisfy two FOCs,

$$
\frac{d\pi_T}{dr} \bigg|_{r=r^*} = -\frac{1}{2(1 + \gamma r^*)^2} \left[ \delta (-1 + \gamma (1 + r^* - w))(1 + p_2^2 - \gamma p_5 r^* + 2\gamma r^{*2} + r^*(-2 + 2\gamma (-1 + w)) - w) \right] = 0,
$$

and

$$
\frac{d\pi_T}{dD_1} \bigg|_{D_1=D_1^*, r=r^*} = -1 + p_1^* + w + \frac{\delta \gamma}{8(-1 + \gamma r^*)^2} \left[ -8\gamma^3 r^{*5} + 3p_2^2 (-1 + \gamma r^*)^3 + 4\gamma^2 r^{*4} (6 + \gamma (-1 + w)) + \gamma r^{*3} (-24 - 12\gamma (-1 + w)) + 19\gamma^2 (-1 + w)^2 + r^{*2} (8 + 12\gamma (-1 + w)) - 25\gamma^2 (-1 + w)^2 - 12\gamma^3 (-1 + w)^3 + 2p_2^2 (-1 + \gamma r^*) (3 + 2\gamma^2 r^{*3}) + \gamma r^{*2} (-4 - 7\gamma (-1 + w)) + 2r^* (1 + 5\gamma (-1 + w) + \gamma^2 (-1 + w)^2) - 3w) r^*(-4 + 5\gamma (-1 + w) + 4\gamma^2 (-1 + w)^2) (-1 + w) + (-1 + w)^2 \right] = 0.
$$
Also,

\[ p^*_2(D^*_1, r^*) = \frac{p^*_2(-1 + \gamma r^*) + (1 + \gamma r^*)(-1 + w)}{-2 + 2\gamma r^*} = p^*_2. \]

Optimal strategies \( r^* \), \( p^*_1 \) and \( p^*_2 \) are determined as

\[ r^* = \sqrt{2s}, \]
\[ p^*_1 = \frac{\sqrt{2s} - 4\gamma s + 2\sqrt{2}(1 - 2\delta)\gamma^2 s^{\frac{3}{2}}}{(1 - \gamma\sqrt{2s})^2}, \]
\[ p^*_2 = \frac{\sqrt{2s} + 2\gamma s}{1 - \gamma\sqrt{2s}}. \]

- Second, we ensure consumers are willing to search in the first period, and in the second period no matter whether the firms they have memberships with are active or not.

\[ E[u_1] = w - p^*_1 + \delta[ - s + \int_{w-r^*}^1 (v^k_2 - p^*_2 + r^*) \, dv^k_2 + (w - r^*)(w - p^*_2)] \]
\[ + (1 - \gamma)(w - p^*_2)] > 0, \]
\[ E[u_{2a}] = - s + \int_{w-r^*}^1 (v^k_2 - p^*_2 + r^*) \, dv^k_2 + (w - r^*)(w - p^*_2) > 0, \]
\[ E[u_{2na}] = w - p^*_2 > 0. \]

The above conditions hold when \( 0 < s \leq \bar{s} = \frac{1}{2(2+\gamma)^2}. \)

- Finally, we discuss sufficiency. Given \( r^* \), \( p^*_1 \) and \( p^*_2 \) as in Proposition 1, we demonstrate that there is only one critical point when \( D^*_1 > 0 \) and \( r^j > 0 \), and it is local maximum.

**Proof of Proposition 2**

**Proof.**

\[ \frac{dr^*}{ds} = \frac{1}{\sqrt{2s}} > 0. \]
\[ \frac{dp^*_2}{ds} = \frac{2 - (1 - \sqrt{2s}\gamma)^2}{(1 - \sqrt{2s}\gamma)^2\sqrt{2s}} > 0. \]
\[ \frac{dp^*_1}{ds} = \frac{-\sqrt{2} + 6\gamma\sqrt{s} + 2(-1 + 2\delta)\gamma^2 (3\sqrt{2} - 2\gamma\sqrt{s})s}{2(-1 + \sqrt{2s}\gamma)^3\sqrt{s}}. \]

As \( \frac{dp^*_1}{ds} \) decreases with \( s \) and \( \frac{dp^*_1}{ds} \big|_{s=\bar{s}} > 0 \), the relationship of \( p^*_1 \) and \( s \) depends on \( \frac{dp^*_1}{ds} \big|_{s=\bar{s}} \).

When \( 0 < \delta \leq \frac{1}{2}, \) or \( \frac{1}{2} < \delta < 1 \) and \( 0 < \gamma \leq \hat{\gamma}, \) the \( p^*_1 \) increases with \( s \); while when \( \frac{1}{2} < \delta < 1 \) and \( \hat{\gamma} < \gamma < 1, \) the \( p^*_1 \) increases then decreases with \( s \) where \( \hat{\gamma} \) is the unique root of equation
\[ \delta \gamma^3 + 3 \delta \gamma^2 - 2 = 0. \]

\[ \frac{d\pi^*_1}{ds} = \frac{1}{\gamma} \frac{dp^*_1}{ds}. \]
\[ \frac{d\pi^*_2}{ds} = \frac{\sqrt{2} - 4 \gamma \sqrt{s} + 14 \sqrt{2} \gamma^2 s - 16 \gamma^3 s^3}{2 \gamma (-1 + \sqrt{2} \gamma)^2 \sqrt{s}} > 0. \]
\[ \frac{d\pi^*_3}{ds} = \frac{1}{2 \gamma (-1 + \sqrt{2} \gamma)^3 \sqrt{s}} \left[ \frac{-\sqrt{2} + 6 \gamma \sqrt{s} - 6 \sqrt{2} \gamma^2 s + 4 \gamma^3 s^2}{(-\sqrt{2} - 6 \gamma \sqrt{s} + 6 \sqrt{2} \gamma^2 s - 36 \gamma^3 s^2 + 16 \sqrt{2} \gamma^4 s^2)} \right] > 0. \]
\[ \frac{d\pi^*_3}{ds} = \gamma N B \frac{d\pi^*_4}{ds} > 0. \]
\[ \frac{d\pi^*_4}{ds} = \frac{M}{(-1 + \sqrt{2} \gamma \sqrt{s})^3 \sqrt{s}} \left[ \frac{-\sqrt{2} + 6 \gamma \sqrt{s} + 6 \sqrt{2} \gamma^2 s - 4 \gamma^3 s^2}{(-\sqrt{2} - 5 \gamma \sqrt{s} + 3 \sqrt{2} \gamma^2 s - 14 \gamma^3 s^2 + 6 \sqrt{2} \gamma^4 s^2)} \right] < 0. \]
\[ \frac{d\pi^*_5}{ds} = \frac{\sqrt{2} - \delta (\sqrt{2} + 2 \gamma \sqrt{s})}{2 \sqrt{s}} < 0. \]

**Proof of Proposition 3**

_Proof._

\[ \frac{dr^*}{d\gamma} = 0. \]
\[ \frac{dp^*_2}{d\gamma} = \frac{4s}{(1 - \sqrt{2} \gamma)^2} > 0. \]
\[ \frac{dp^*_3}{d\gamma} = \frac{8 \sqrt{2} s \delta \gamma s}{(-1 + \sqrt{2} \gamma)^3} < 0. \]
\[ \frac{d\pi^*_1}{d\gamma} = \frac{\sqrt{2} s - 6 \gamma s + 2 \sqrt{2} \gamma (3 + 2 \delta) \gamma^2 s + 4 (-1 + 2 \delta) \gamma^3 s^2}{\gamma^2 (-1 + \sqrt{2} \gamma)^3} < 0. \]
\[ \frac{d\pi^*_2}{d\gamma} = \frac{-\sqrt{2} s + 4 \gamma s + 2 \sqrt{2} \gamma^2 s}{\gamma^2 (-1 + \sqrt{2} \gamma)^2} < 0. \]
\[ \frac{d\pi^*_3}{d\gamma} = \frac{\frac{d\pi^*_1}{d\gamma} + \delta \frac{d\pi^*_2}{d\gamma}}{d\gamma} < 0. \]
\[ \frac{d\pi^*_4}{d\gamma} = \frac{-N (8 \delta \gamma^2 (-3 + \sqrt{2} \gamma) s^2}{(-1 + \sqrt{2} \gamma)^3} < 0. \]
\[ \frac{dTS}{d\gamma} = -N B \delta s < 0. \]

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\[ \frac{dCS}{d\gamma} = M\delta s(1 - 3\sqrt{2}s\gamma - 18\gamma^2 s + 6\sqrt{2}s\gamma^3 s) \left( -1 + \sqrt{2}s\gamma \right)^3 \]

increases with \( \gamma \), and \( \frac{dCS}{d\gamma} \bigg|_{\gamma=0} < 0 \). When \( 0 < s < \frac{1}{18} \), \( 0 < \gamma < 1 \); while when \( \frac{1}{18} < s < \frac{1}{8} \), \( 0 < \gamma < \frac{\sqrt{2} - 4\sqrt{s}}{2\sqrt{s}} \). The relationship of \( CS \) on \( \gamma \) depends on \( \frac{dCS}{d\gamma} \bigg|_{\gamma=\min\{1, \frac{\sqrt{2} - 4\sqrt{s}}{2\sqrt{s}}\}} \). Consumer surplus decreases when \( 0 < s \leq s_1 \) or \( s_2 \leq s < \frac{1}{8} \), but shows a decrease followed by an increase when \( s_1 < s < s_2 \), where the threshold \( s_1 \) and \( s_2 \) are uniquely determined by the equation \( 1 - 3\sqrt{2}s - 18s + 6\sqrt{2}s^2 = 0 \) and \( \sqrt{2} - 6\sqrt{s} + 12s^2 = 0 \) respectively, both of which fall within \([0, \frac{1}{8}]\).

**Proof of Proposition 4**

*Proof.*

\[ p^*_1 - p^*_1 - p^*_B = \frac{4\sqrt{2}\gamma s^2}{(-1 + \sqrt{2}s\gamma)^2} < 0. \]
\[ p^*_2 - p^*_2 - p^*_B = \frac{4\gamma s}{1 - \sqrt{2}s\gamma} > 0. \]
\[ p^*_2 - r^* - p^*_B = \frac{\sqrt{2}s - 6\gamma s}{-1 + \sqrt{2}s\gamma} < 0. \]
\[ \pi^*_1 - \pi^*_1 - \pi^*_B = \frac{1}{\gamma}(p^*_1 - p^*_B) < 0. \]
\[ \pi^*_2 - \pi^*_2 - \pi^*_B = \frac{4\sqrt{2}s\gamma s}{1 - \sqrt{2}s\gamma} > 0. \]
\[ \pi^*_T - \pi^*_T - \pi^*_B = \frac{-8\delta\gamma^2 s^2}{(-1 + \sqrt{2}s\gamma)^2} < 0. \]
\[ \pi^*_I - \pi^*_I - \pi^*_B = \frac{-8\delta\gamma^3 s^2 NB}{(-1 + \sqrt{2}s\gamma)^2} < 0. \]
\[ CS - CSB = \frac{\delta\gamma s(-1 + 2\sqrt{2}s\gamma + 6\gamma^2 s)M}{(-1 + \sqrt{2}s\gamma)^2} < 0. \]
\[ TS - TS^B = -\delta\gamma s NB < 0. \]
References


