Practical Finance: An Approximate Solution to Lifecycle Portfolio Choice

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Abstract
Research on normative household financial choices almost always gives quantitative guidance in one of two forms: a simple formula for an unrealistic setting, or numerical solutions for only a few parameter sets in a realistic setting. We propose a middle-ground approach we call *practical finance*: analytic approximations to optimal solutions as a function of relevant parameters in realistic settings that are easily computed in a spreadsheet. We provide such an approximation for lifecycle portfolio choice with labor income. Across 5,103 parameter sets, our approximation results in welfare that is on average only 0.06% lower than that of the optimal solution.

Keywords: Portfolio choice, personal finance, financial advice

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If you need medical treatment, you should consult a physician. If you need your car fixed, you should see an auto mechanic. If you need help managing your personal finances, should you get advice from a research economist?

Academic economics research offers plenty of good qualitative advice. Diversify. Choose low-cost index funds. All else equal, smoother consumption over time is desirable. But ultimately, individuals must make *quantitative* choices—what percent of their portfolio to allocate to each asset, how much of their income to save, etc. Here, guidance from academic economics research too often falls into one of two categories: a simple analytic formula for an unrealistic setting, or a numerical solution computed for only a few parameter sets in a realistic setting.

The analytic formula provides an easy path to a solution, but it is the correct solution for circumstances that often bear little resemblance to your life.[1] The numerical solution may be for a setting that qualitatively matches your circumstances, but the parameter values for which the authors provide solutions are usually quite different from your own. Your only choices are to implement as-is one of the mismatched solutions that are provided; make ad hoc adjustments to one of the solutions; ignore the solutions and resort to decision rules not based on academic research; or write your own code to obtain solutions that *are* appropriate for you, which is prohibitively costly even for professional economists.

We propose that a middle-ground research approach should become more common, a style that we dub *practical finance*: research that provides an analytic approximation to the optimal solution in a realistic setting as a function of the relevant parameters that is easily computed in a spreadsheet, so that readers can obtain the solution that applies to their own parameter values. Although the research output could in principle be an atheoretical Taylor approximation, such an approximation would be less useful because it would be unpersuasive to the intended user. Ideally, the approximation would be structured to provide some economic intuition for its result.

In this paper, we construct such an approximation for optimal lifecycle asset allocation

[1] The best-known formula for the optimal risky portfolio share is found in Merton (1969), but it is for the case where the investor has no future labor income.

between the stock market and a risk-free asset when labor supply is fixed, labor income is risky and non-tradable, and the investor has constant relative risk aversion (CRRA) utility. Cocco, Gomes, and Maenhout (2005), referred to as CGM hereafter, is the canonical paper that numerically solves for optimal portfolio policies in this setting. Despite the large scholarly impact of CGM, their solutions have had very little effect on financial advice in practice (Choi, 2022), perhaps due to the complexity of the mathematics and the inability for a reader to obtain solutions for different parameters without re-running the authors' code.

We provide an approximation to the CGM solution that is structured around the Merton (1971) and Bodie, Merton, and Samuelson (1992) formulas for the case where labor supply is fixed and labor income is riskless.

Let α^* be a CRRA investor's optimal risky portfolio share if he had no labor income (Merton, 1969; Restoy, 1992; Campbell and Viceira, 1999): the equity premium divided by the product of relative risk aversion and the variance of the stock market's log return. With risk-free labor income, the investor should still invest a proportion α^* of his wealth in the stock market, but his wealth is not just his financial wealth W, but also the present discounted value of his labor income H (human capital). Therefore, $\alpha^*(W + H)$ dollars should be invested in the stock market. Because riskless human capital is an implicit risk-free bond holding, $\alpha^*(1 + H/W)$ fraction of his financial wealth should be invested in the stock market in order to get $\alpha^*(W + H)$ dollars of exposure to the stock market. To calculate the value of H, discount future labor income using the risk-free interest rate (Merton, 1971; Bodie, Merton, and Samuelson, 1992).

If labor income is risky yet uncorrelated with stock returns—as is the case empirically for the average household (Gomes, Haliassos, and Ramadorai, 2021)[2]—CGM find that human capital still behaves like an implicit fixed income position. But if we were to try to allocate $\alpha^*(1 + H/W)$ fraction of the financial portfolio to the stock market in this case, the discount rate that should be used to compute H for the formula to deliver the correct solution becomes unclear.[3]

[2] Wuthenow (2024) estimates using the Panel Study of Income Dynamics that from 1980 to 2020, the correlation between real labor income growth and the S&P 500's real returns for the average household is 0.007.

[3] Jorgensen and Fraumeni (1989), Winter, Schlafmann, and Rodepeter (2012), Love (2013) and Gomes and Smirnova

Our efforts will concentrate on providing approximately correct discount rates for expected future labor income as a function of the important model parameters.

We begin by numerically solving the CGM model for 5,103 different parameter sets, independently varying relative risk aversion, the variance of permanent labor income shocks, the variance of temporary labor income shocks, the expected labor income path as a function of age, the fraction of final working-life income that is replaced by Social Security benefits, the risk-free rate, and the equity risk premium. For each age within each parameter set, we obtain the optimal asset allocation for many different values of current financial wealth W, imposing no-leverage and no-short-sale constraints. We then estimate the discount rates applied to future expected labor income that results in the H that minimizes the mean squared distance between the numerical solutions and the corresponding solutions produced by $\max\{0, \min\{1, \alpha^*(1 + H/W)\}\}$.

More specifically, for a given parameter set, we begin by estimating the one-year-ahead discount rate $r_{y,99}$ that applies to expected age 100 income from the perspective of the age 99 self. We then hold fixed $r_{y,99}$, and we estimate the one-year-ahead discount rate $r_{y,98}$. The age 98 self divides expected age 99 income by $1 + r_{y,98}$ to obtain its present value, and divides expected age 100 income by $(1 + r_{y,98})(1 + r_{y,99})$ to obtain its present value. We repeat this process of estimating one-year-ahead discount rates, moving one year younger each time until we have obtained $r_{y,22}$, the one-year-ahead discount rate at age 22. The value of human capital at age 22, H_{22}, is the sum of discounted expected future labor incomes $Y_{23}, Y_{24}, \ldots, Y_{100}$ (conditional on the age-22 information set and conditional on still being alive at the future age):

$$H_{22} = \frac{E_{22}(Y_{23})}{1 + r_{y,22}} + \frac{E_{22}(Y_{24})}{(1 + r_{y,22})(1 + r_{y,23})} + \cdots + \frac{E_{22}(Y_{100})}{(1 + r_{y,22})(1 + r_{y,23}) \ldots (1 + r_{y,99})} \quad (1)$$

We find that the one-year-ahead discount rate for labor income tends to be highest in early life,

(2023) discount risky future labor income at the risk-free interest rate. Graham and Webb (1979) and Eisner (1980) discount by the average of the return on personal savings and the consumer loan interest rate. Huggett and Kaplan (2016) discount using their model agent's stochastic discount factor.

decreases until mid-life to a level substantially above the risk-free rate, and drops discretely to nearly the risk-free rate at retirement, when the retirement income benefit is risk-free.

Our final step develops simple functions of the model parameters (relative risk aversion, log equity premium, age, etc.) that approximate the one-year-ahead discount rate curves. These are the functions that a real-life household could use to estimate its own discount rates. We create these approximations by regressing one-year-ahead discount rates on the model parameters.

To test the goodness of our approximation, we compute portfolio equity shares at every point in our discretized state space grid from the formula $\max\{0, \min\{1, \alpha^*(1 + H/W)\}\}$, using our approximate discount rates to calculate H. We regress actual optimal equity shares on their corresponding approximately optimal equity shares. The resulting R^2 is 0.99, with a slope of 0.95 and an intercept of 0.05. This tight regression fit indicates that it is possible to obtain quite accurate approximations of optimal asset allocations using simple functions. The root mean square approximation error across all parameter sets and grid points is 3.66 percentage points.

The above evaluation of our approximation's accuracy equally weights all points in our state space grid, regardless of their likelihood of occurring and approximation error's impact on welfare at each point. As an alternative, we compute the expected discounted lifetime utility for a 22-year-old whose only financial wealth is the labor income she earned in the current period and who follows our approximate asset allocation policy. Across the 5,103 parameter sets, the average discounted expected utility loss relative to following the optimal asset allocation policy is equivalent to only a 0.06% reduction in lifetime consumption. The analogous loss from following the common "set your equity share to 100 minus your age" rule of thumb is 2.00%. A constant 60% stock portfolio allocation results in an average welfare loss of 3.75%.

In an extension, we consider the case where labor income shocks are correlated with stock market returns. We find that using the formula $\max\left\{0, \min\left\{1, \alpha^*(1 + H/W) - \beta_{perm} H/W\right\}\right\}$ during working life performs remarkably well, where β_{perm} is the coefficient from regressing log permanent income shocks on log stock market returns, and H is computed using the same discount rates as in the uncorrelated case. Correlations of temporary income shocks with stock market returns do not have a material impact on the optimal asset allocation.

Our paper adds to a small body of academic literature that can be classified as "practical finance," although many of these previous papers stop at providing easily computable rules of thumb without allowing these rules to vary with model parameters. Winter, Schlafmann, and Rodepeter (2012) evaluate the performance of three consumption rules of thumb. Love (2013) estimates, for one set of parameters, optimal rules of thumb for asset allocation and savings rates that are linear in either age or the ratio of financial wealth to total (human plus financial) wealth. Fischer and Koch (2025) provide approximations to optimal asset allocations that are linear in age and the ratio of current labor income to financial wealth for a small number of parameter sets. Brown, Cederburg, and O'Doherty (2017) propose a rule of thumb for optimally allocating assets between traditional and Roth tax-advantaged retirement savings accounts. Agarwal, Driscoll, and Laibson (2013) derive an analytic approximation to the optimal fixed-rate mortgage refinancing strategy.

Our work also contributes to the literature on optimal lifecycle asset allocation in the presence of risky labor income. Merton (1971) and Svensson and Werner (1993) provide solutions for agents with constant absolute risk aversion utility. He and Pearson (1991), He and Pagés (1993), Cuoco (1997), and El Karoui and Jeanblanc-Picqué (1998) consider settings where labor income risk is spanned by tradable asset risk. Duffie et al. (1997), Koo (1998, 1999), and Viceira (2001) study infinitely lived agents with stationary labor income growth processes that are not perfectly correlated with traded asset returns. While the analytic results in these works offer valuable insights, they are challenging to use for direct quantitative guidance in real-life financial decisions.

Besides CGM, numerical solutions for settings with realistic uninsurable labor income processes and other complicating factors are provided, for example, by Cocco (2005), Gomes and Michaelides (2005), Yao and Zhang (2005), Davis, Kubler, and Willen (2006), Benzoni, Collin-Dufresne, and Goldstein (2007), Gomes, Kotlikoff, and Viceira (2008), Gomes, Michaelides, and Polkovnichenko (2009), Wachter and Yogo (2010), Campanale, Fugazza, and Gomes (2015), Huggett and Kaplan (2016), Fagereng, Gottlieb, and Guiso (2017), Dahlquist, Setty, and Vestman

(2018), Catherine (2022), and Duarte et al. (2024).

The rest of the paper is organized as follows. Section 1 describes the portfolio choice problem setup and its numerical solutions. Section 2 calculates the discount rates for human capital over the life cycle. Section 3 approximates the discount rates with regressions and uses the fitted discount rates to calculate approximately optimal portfolio allocations. Section 4 compares the utility cost of different portfolio choice rules. Section 5 presents the extension where permanent labor income shocks are correlated with stock market returns, and Section 6 concludes.

1 The Portfolio Choice Problem and Its Solutions

1.1 Problem Setup (Cocco, Gomes, and Maenhout, 2005)

Let t denote the agent's age. The agent starts working life at age t_0, lives to a maximum possible age of T, retires at the end of a fixed and exogenous age $K \leq T$, and at each t maximizes the discounted sum of CRRA utility, adjusted for stochastic mortality:

$$E_t \sum_{\tau=t}^{T} \delta^{\tau-t} \left(\prod_{j=t}^{\tau-1} p_j \right) \frac{C_\tau^{1-\gamma}}{1-\gamma} \tag{2}$$

where $0 < \delta < 1$ is the discount factor, C_τ is age τ consumption, γ is the coefficient of relative risk aversion, and p_τ is the probability that the agent is alive at age $\tau + 1$ conditional on being alive at age τ.

Taxes are not explicitly modeled, so all variables are implicitly after-tax where relevant. During working life, log real labor income is the sum of a deterministic cubic function of age $f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$, a permanent shock v_t, and a temporary shock ε_t:

$$\log(Y_t) = f(t) + v_t + \varepsilon_t \quad \text{for } t \leq K \tag{3}$$

The permanent shock v_t follows the random walk process $v_t = v_{t-1} + u_t$, where

$u_t \sim N(0, \sigma_u^2)$ is independently and identically distributed (i.i.d.) each period. The temporary shock $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ is i.i.d. and uncorrelated with the permanent shock. Define permanent income during working life as labor income excluding its temporary component: $P_t \equiv \exp(f(t) + \nu_t)$.

In retirement, the real retirement income benefit is a fixed fraction λ of permanent real labor income in the final year of working life[4]:

$$Y_t = \lambda P_K \quad \text{for } K < t \leq T \tag{4}$$

We define permanent income in retirement to be the retirement income benefit.

There are two financial assets available: a risk-free asset and a risky asset (the stock market). Every period, the risk-free asset yields a fixed real gross return $R_f > 0$ whose log is r_f. The risky asset's real gross return $R_t > 0$ is drawn from a lognormal distribution, so that $r_t \equiv \log(R_t) \sim N(r, \sigma_r^2)$. We assume in the baseline case that the risky asset return is uncorrelated with labor income shocks.

The agent enters each age *t* with real financial assets worth F_t and receives real labor income Y_t. Define $X_t \equiv F_t + Y_t$ as cash-on-hand at *t*. The agent decides how much to consume and the fraction α_t of the remaining financial portfolio $W_t \equiv X_t - C_t$ to invest in the risky asset. The agent cannot borrow or sell short. Thus, the agent is subject to the constraints

$$X_{t+1} = \underbrace{\overbrace{(X_t - C_t)}^{W_t} \left(\alpha_t R_{t+1} + (1 - \alpha_t) R_f\right)}_{F_{t+1}} + Y_{t+1} \tag{5}$$

$$0 < C_t \leq X_t \tag{6}$$

$$0 \leq \alpha_t \leq 1 \tag{7}$$

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4 In the U.S. Social Security system, the retirement income benefit is a complicated nonlinear function of 35 years of labor income. Setting retirement income to depend only on final working-life permanent income allows us to not keep track of the history of labor income as a state variable.

Because the value function is homogeneous with respect to current permanent labor income[5], we can drop permanent labor income as a state variable and replace the state variable X_t with the normalized variable $x_t \equiv X_t/P_t$. Consumption is solved for as a multiple of permanent income, $c_t \equiv C_t/P_t$. Let $w_t \equiv x_t - c_t$.

We solve this problem numerically at each grid point of a discretized state space. The 201 normalized cash-on-hand values on the grid range from 0.25 to 50. Details of the numerical solution technique are given in the Appendix.

1.2 Parameterization

Agents start their economic life at age 22, work until the end of age 66 (the current Social Security full retirement age is 67), and live no longer than age 100. Mortality probabilities at each age are set to match the life table for the total U.S. population from the National Center for Health Statistics (NCHS). Following CGM, the time discount factor δ is set to 0.96. The standard deviation of the risky asset's log return σ_r is set to 0.185, the annualized standard deviation of the CRSP value-weighted market index log monthly return in excess of the 1-month Treasury log return from July 1926 to July 2024.

We solve the problem for a range of values of the parameter vector $\Omega = (\gamma, \sigma_u, \sigma_\epsilon, a_0, a_1, a_2, a_3, \lambda, r_f, r - r_f)$ that covers much of the space that is likely to be economically relevant.

- Relative risk aversion γ: 4, 5, 6, 7, 8, 9, and 10. Risk aversion of 10 is commonly regarded as the upper limit of reasonable risk aversion (Mehra and Prescott, 1985). We find that at a risk aversion of 4, the optimal equity allocation is very frequently at the 100% upper boundary. Therefore, the optimal equity allocation for lower risk aversions will be even more uniformly 100%, making any approximation that weakly decreases with risk aversion and is accurate for risk aversions above 4 able to accurately fit solutions for risk

[5] See Carroll (2022, pp. 5-6) for a proof of this homogeneity.

aversions below 4.

- Standard deviation of log permanent income shocks σ_u: 0.102, 0.103, and 0.130. These are CGM's estimates for individuals with no high school, only a high school, and a college education, respectively.
- Standard deviation of log temporary income shocks σ_ε: 0.242, 0.272, and 0.325. These are CGM's estimates for individuals with a college, only a high school, or no high school education, respectively.
- Polynomial coefficients for the deterministic portion of the real log labor income path: $\{a_0, a_1, a_2, a_3\}$ = {-2.1361, 0.1684, -0.00353, 0.000023}, {-2.1700, 0.1682, -0.00323, 0.000020}, {-4.3148, 0.3194, -0.00577, 0.000033}. These are CGM's estimates for individuals with no high school degree, only a high school degree, and a college degree, respectively. As education increases, income grows more steeply in early life and is higher on average.
- Retirement income benefit replacement rate λ: 0.4, 0.6, 0.8. Biggs and Springstead (2008) estimate that for a 64-66 year old beneficiary, the median Social Security benefit as a percent of the individual's average inflation-adjusted labor income over the final five years prior to claiming benefits ranges from 40% in the highest lifetime earnings quintile to 82% in the second-lowest lifetime earnings quintile.[6]
- Log real risk-free rate r_f: 0, 0.01, and 0.02.
- Log risk premium of the risky asset $r - r_f$: 0.02, 0.03, and 0.04. These are considerably lower than the 6.5% log equity premium realized from July 1926 to July 2024. However, even a 2% log equity premium results in optimal equity allocations that are frequently 100%. Therefore, any approximation that accurately fits the solution for log equity premia of 4% or less should be accurate for log equity premia above 4%.

[6] The median replacement rate is infinite for the lowest lifetime earnings quintile because the median individual in that quintile had no labor income in the five years prior to claiming benefits.

Allowing each of these parameters to vary independently of the others creates 7 × 3 × 3 × 3 × 3 × 3 × 3 = 5,103 different pre-retirement parameter sets for which we solve the problem. In retirement, labor income (the retirement income benefit) is risk-free and constant over time; the parameters governing working-life labor income and the retirement income replacement rate become irrelevant, leaving only 7 × 3 × 3 = 63 different parameter sets to consider.

1.3 Solutions

Figure 1 plots, for a particular set of parameter values, the percent of the financial portfolio allocated to the risky asset at various ages as a function of normalized invested wealth w_t (cash-on-hand at the beginning of the period minus current consumption, normalized by current permanent income). The investor in the graph has a relative risk aversion of 7 and faces the labor income process of a college graduate, a log risk-free rate of 2%, a log equity premium of 2%, and a retirement income benefit replacement rate of 40%. The risk-free rate approximately equals the five-year TIPS real yield in 2024. The equity premium is similar to the 1.9% forecast for U.S. large-cap equities' log excess return over cash made by the hedge fund AQR at year-end 2023 using dividend and payout discount models (Portfolio Solutions Group, 2024).

If there is no labor income, the approximately optimal portfolio equity share is given by an analogue to the Merton (1969) continuous time solution that applies in a discrete time setting (Restoy, 1992; Campbell and Viceira, 1999), subject to leverage and short-sale constraints:

$$\alpha^* = \max\left\{0, \min\left\{1, \frac{r - r_f + \frac{1}{2}\sigma_r^2}{\gamma\sigma_r^2}\right\}\right\} \tag{8}$$

For the parameter values described above, $\alpha^* = 0.155$, which is depicted by the horizontal solid gray line in Figure 1.

The optimal allocation at age 99 also has an approximate analytic characterization. At this age, there is only one remaining period of labor income, and conditional on remaining alive, the labor income is risk-free. The possibility of dying before age 100 does not alter the optimal asset allocation choice at age 99, which only affects utility if the agent survives to age 100. Therefore, the approximately optimal equity allocation is given by $\alpha_{99} = \max\{0, \min\{1, \alpha^*(1 + H_{99}/W_{99})\}\} = \max\{0, \min\{1, \alpha^*(1 + h_{99}/w_{99})\}\}$, where H_{99} is age-100 labor income (conditional on being alive at age 100) discounted at the risk-free interest rate, and h_{99} is H_{99} divided by age-99 permanent income. As w_{99} approaches infinity, α_{99} asymptotes to α^*, since the economic importance of next period's labor income becomes negligible. As w_{99} approaches zero, α_{99} rises to 100%, since an all-equity allocation in the financial portfolio constitutes a negligible amount of risk relative to remaining lifetime resources. We see these patterns in the black dotted line in Figure 1.

At age 75, the present discounted value of riskless retirement benefit payments is much higher than it is at age 99 because there are more such payments in expectation. Correspondingly, the optimal asset allocation for a given amount of invested financial wealth is much more aggressive at age 75. Whereas a 99-year-old who is investing financial wealth equal to ten times her annual Social Security benefit should only allocate 17% to stocks, a 75-year-old who is investing the same amount should allocate 44% to stocks. Like at age 99, α at age 75 starts at 100% for low levels of invested financial wealth and declines towards the Merton asymptote α^* as invested financial wealth increases.

Before retirement, the relationship between invested financial wealth and optimal equity share is less straightforward to intuit; since working-life labor income is risky, it is unclear whether human capital should be treated more like a risk-free asset or a risky asset. We see in Figure 1 that qualitatively, the shape of working-life α with respect to invested financial wealth is the same as in retirement, starting at 100% for low levels of wealth and asymptotically declining towards the Merton solution α^* as wealth increases. Therefore, risky human capital whose growth is uncorrelated with the stock market's return is akin to a fixed income holding.

Even with relatively high risk aversion and an equity premium that is much lower than its historical realization, a 45-year-old optimally holds 100% of her portfolio in stocks as long as her financial wealth to be invested is less than 1.6 times her permanent income. Common financial advice is to hold any money you might spend in the near term entirely in cash, and of the remaining money, allocate a percentage equal to one hundred minus your age to stocks (Choi, 2022). The CGM model does not recommend that this 45-year-old allocate 55% to stocks until invested wealth reaches 3.8 times her permanent income. The average target date retirement fund intended for somebody 20 years from retirement holds about 75% of its assets in stocks (Shoven and Walton, 2021), which the CGM model recommends for this individual when invested wealth equals 2.4 times permanent income.

2 Labor Income Discount Rates

2.1 Methodology

We conjecture that even before age 99, the optimal equity share can be well-approximated by the expression

$$\hat{\alpha}_t = \max\{0, \min\{1, \alpha^*(1 + h_t/w_t)\}\} \tag{9}$$

given appropriate discount rates to compute h_t, the present discounted value of normalized future labor income. This is not a vacuous conjecture because h_t is not allowed to vary with normalized invested wealth w_t and the functional form constrains the shape of the relationship between $\hat{\alpha}_t$ and w_t.

Fix a value of the parameter vector Ω. Our procedure for finding discount rates for an agent with these parameter values starts from the perspective of the agent at age 99. At this age, the only future labor income comes at age 100, Y_{100}. The value of p_{99} (the probability of dying before age 100) does not alter the optimal asset allocation choice at age 99. We calculate the value of human capital normalized by age-99 permanent income as

$$h_{99} = \frac{E_{99}(Y_{100}/P_{99})}{1 + r_{y,99}} \tag{10}$$

Here and subsequently, any expectations operator applied to labor income realized at age *t* conditions on surviving until *t*, although we suppress the conditioning notation.

Let *i* index the normalized cash-on-hand grid points x_i in our discretized state space from 1 to *I*. We have already numerically calculated the optimal equity share $\alpha_{99,i}$ and normalized consumption $c_{99,i}$ for each value of x_i. We now search for the discount rate $r_{y,99}$ that minimizes the sum of squared distances between our approximations $\hat{\alpha}_{99,i}$ and the true values $\alpha_{99,i}$ across all x_i.

$$r_{y,99} = \text{argmin}_r \sum_{i=1}^{I} \left(\hat{\alpha}_{99,i} - \alpha_{99,i}\right)^2 \tag{11}$$

$$\hat{\alpha}_{99,i} = \max\left\{0, \min\left\{1, \alpha^* \left(1 + \frac{E_{99}(Y_{100}/P_{99})/(1+r))}{x_i - c_{99,i}}\right)\right\}\right\} \tag{12}$$

We next search for the discount rate to apply at age 98. At this age, there are two future income arrivals to discount in order to compute h_{98}. We discount age-99 income by $1 + r_{y,98}$, and age-100 income by $(1 + r_{y,98})(1 + r_{y,99})$, holding $r_{y,99}$ fixed at the value we identified in equation (11). Thus, we search for the value of the one-period-ahead discount rate $r_{y,98}$ that minimizes the sum of squared distances between $\hat{\alpha}_{98,i}$ and $\alpha_{98,i}$:

$$r_{y,98} = \text{argmin}_r \sum_{i=1}^{I} \left(\hat{\alpha}_{98,i} - \alpha_{98,i}\right)^2 \tag{13}$$

$$\hat{\alpha}_{98,i} = \max\left\{0, \min\left\{1, \alpha^* \left(1 + \frac{\frac{E_{98}(Y_{99}/P_{98})}{1+r} + \frac{E_{98}(Y_{100}/P_{98})}{(1+r)(1+r_{y,99})}}{x_i - c_{98,i}}\right)\right\}\right\} \tag{14}$$

We continue this procedure, reducing age by one year at a time and holding fixed all discount rate values computed in previous steps. For age *t*, we have:

$$r_{y,t} = \text{argmin}_r \sum_{i=1}^{I} \left(\hat{\alpha}_{t,i} - \alpha_{t,i}\right)^2 \tag{15}$$

$$\hat{\alpha}_{t,i} = \max\left\{0, \min\left\{1, \alpha^* \left(1 + \frac{\sum_{j=t+1}^{100} \frac{E_t\left(Y_j/P_t\right)}{(1+r)\prod_{m=t+1}^{j-1}\left(1+r_{y,m}\right)}}{x_i - c_{t,i}}\right)\right\}\right\} \tag{16}$$

Once in retirement, the value of normalized labor income is non-stochastic and always equal to 1, since permanent income in retirement is defined as the size of the retirement income benefit. Before retirement at the end of age K, there is uncertainty about income both during working life and during retirement. The expectations at $t < K$ of normalized income earned at $t + n$ conditional on surviving to $t + n$ are

$$E_t(Y_{t+n}/P_t) = \exp\left(f(t+n) - f(t) + \frac{1}{2}(n\sigma_u^2 + \sigma_\epsilon^2)\right) \quad \text{if } t+n \leq K \tag{17}$$

$$E_t(Y_{t+n}/P_t) = \lambda \exp\left(f(K) - f(t) + \frac{1}{2}(K-t)\sigma_u^2\right) \quad \text{if } t+n > K \tag{18}$$

2.2 Examples of Labor Income Discount Rates over the Life Cycle

Figure 2 plots the life-cycle path of one-period-ahead labor income discount rates for relative risk aversion of 4, 7, or 10, holding all other parameters fixed at a particular vector of values (labor income process of a college graduate, a log risk-free rate of 2%, a log equity premium of 2%, and a retirement income benefit replacement rate of 40%).

Higher values of risk aversion are associated with higher discount rates during working life.

Discount rates are highest when young because of liquidity constraints created by rising expected labor income during the first part of working life.[7] In the second half of working life, discount rates are roughly flat with respect to age unless risk aversion is high, in which case discount rates rise with age.

During retirement, discount rates do not vary with risk aversion, which makes sense because labor income in the form of the retirement benefit is risk-free. The discount rate is a bit above the risk-free rate and slightly hump-shaped with respect to age. Merton (1971) and Bodie, Merton, and Samuelson (1992) show analytically that in a continuous time setting with no borrowing constraints, risk-free labor income is discounted at the risk-free rate when constructing optimal portfolios. The fact that our discount rates do not exactly equal the risk-free rate is attributable to the difference between continuous time and our discrete time setting where the time unit is one year.[8]

Figure 3 shows that higher retirement income benefit replacement rates do not materially affect early-life discount rates but make the rise in discount rates just before retirement more pronounced. A higher replacement rate increases the value of human capital, which makes optimal financial portfolios more aggressive. But higher discount rates on labor income decrease the value of human capital, making optimal financial portfolios more conservative. Therefore, the fact that discount rates rise with the replacement rate indicates that as retirement income increases relative to working-life income, financial risk-taking does not rise as quickly as it would if the discount rate remained unchanged.

In Figure 4, we display the path of discount rates when the risk-free interest rate varies. As is intuitive, an increase in the risk-free rate increases labor income discount rates about one-for-one. Figure 5 illustrates that in contrast, an increase in the equity premium decreases labor income

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[7] Online Appendix Figure 1 shows that when the deterministic portion of log labor income is constant during working life, discount rates are flat during the first half of working life. Discount rates in the second half of working life are not materially altered by switching the deterministic portion of working-life log labor income to a constant value.

[8] The presence of borrowing constraints in our setting may play a role as well, but we see that even at age 99, when the portfolio problem reduces to a one-period problem and borrowing constraints become irrelevant, our discount rates diverge from the risk-free rate.

discount rates.

3 Low-Dimensional Approximations of Optimal Life-Cycle Portfolio Choices

3.1 Discount Rate Approximation Method

Having obtained one-period-ahead discount rates for each age and parameter set, we now create low-dimensional approximations of these discount rates using regressions of discount rates on model parameters.

In retirement, the variance of labor income shocks during working life is irrelevant to the optimal asset allocation, as is the fraction of working life income replaced by the retirement income benefit. Therefore, we run two separate regressions: one for discount rates when the next period is in working life, and one for discount rates when the next period is in retirement. Visual inspection of Figures 2-5 reinforces the case for using separate approximations.

For discount rates during working life, we regress discount rates on relative risk aversion, log equity premium, log risk-free rate, variance of the permanent income shock, variance of the temporary income shock, retirement income replacement rate, and in some specifications either age or a quadratic in age. We have 5,103 sets of parameter values and 44 years in which the next year is within working life, so there are 5,103 × 44 = 224,532 discount rate observations in this regression. For years where the following year is a retirement year, we regress discount rates on relative risk aversion, log equity premium, log risk-free rate, and in some specifications either age or a quadratic in age. Due to the smaller number of parameters that are relevant during retirement, there are only 63 sets of parameter values, so there are 63 × 34 years = 2,142 discount rate observations in the retirement regression. We also try a specification in the working-life and retirement regressions where age and the square of age are interacted with each of the other explanatory variables.

3.2 Approximation Coefficients

Table 1 shows coefficients from the working-life discount rate regressions. The first column excludes age from the set of explanatory variables. With this simple form, we are already able to achieve an adjusted R^2 of 0.864. Although this will not be our main specification, it may be appealing to many users because the approximate discount rate that comes out of this specification does not change with age within working life—that is, the term structure is flat. Nearly all finance students are accustomed to computing present values under a flat term structure, so that a cashflow occurring τ periods in the future is discounted by $(1+r)^{\tau}$. Many students may find it challenging to compute present values when the one-period-ahead discount rate varies with each age.

Consistent with the visual evidence from Figures 2 to 5, the discount rate is increasing in risk aversion, the risk-free rate, and the retirement income replacement rate, and decreasing in the equity premium. When risk aversion rises by 1, the discount rate rises by 0.9 percentage points. A 1 percentage point rise in the log risk-free rate results in a 1.1 percentage point rise in the discount rate. The other two parameters have smaller effects. A 1 percentage point increase in the log equity premium decreases the discount rate by 0.3 percentage points. A 10 percentage point increase in the retirement income replacement rate raises the discount rate by only 0.1 percentage points.

In addition, the discount rate increases with the riskiness of labor income. Permanent income shock risk has a much larger effect than temporary income shock risk. Going from the permanent income shock variance of a high school dropout to a college graduate increases the discount rate by 2.8 percentage points. Going from the temporary income shock variance of a high school dropout to a college graduate decreases the discount rate by a negligible 0.1 percentage points.

The second column adds age as an explanatory variable. An additional 10 years of age decreases the discount rate by 0.3 percentage points. The addition of this explanatory variable raises the adjusted R^2 from 0.864 to 0.880. The coefficients on the other explanatory variables remain unchanged relative to the first column because all model parameters are uncorrelated with each other in the data we created to feed into the regression.

The third column adds the square of age as an explanatory variable. This quadratic term is estimated to have a positive coefficient, consistent with the second derivative of the discount rate with respect to age being positive in Figures 2 to 5. With the square of age in the regression, the adjusted R^2 increases to 0.886.

The fourth column contains coefficients from a regression where we add interactions of all non-age variables with age and the square of age. (The coefficients on the interactions are not shown.) Despite twelve additional explanatory variables, the adjusted R^2 rises only to 0.907. We judge this incremental R^2 gain to be insufficient to justify the additional complexity created for the end user. Therefore, we adopt the third column's specification as our approximation going forward.

Table 2 shows discount rate regressions for the retirement period. The first column excludes age from the set of explanatory variables. Risk aversion has nearly no effect on the discount rate for the risk-free retirement benefit. The effects of the equity premium and risk-free rate have the same signs as during working life, but their magnitudes are slightly attenuated. The adjusted R^2 in this specification is 0.721. Adopting the approximations that come from the first columns of Tables 1 and 2 allows users to compute the value of human capital while changing the one-period-ahead discount rate only once—at the retirement threshold.

Adding age as an explanatory variable in the second column of Table 2 raises the adjusted R^2 to 0.736, while adding the square of age in the third column substantially improves the adjusted R^2 to 0.819 by capturing the hump shape of discount rates with respect to age in retirement. The adjusted R^2 increase to 0.841 in the fourth column, which adds as explanatory variables interactions of age and age squared with all the non-age variables. Given the modest improvement in fit relative to the number of additional explanatory variables, and to maintain consistency with our main specification during working life, we adopt the specification in the third column for our approximation of discount rates in retirement.

3.3 Calculation Example

Suppose we are calculating the optimal equity allocation for a 55-year-old with relative risk aversion of 7, a log real risk-free rate of 2%, a log equity premium of 2% (which translates into a $\exp(0.02 + 0.02 + 0.5 \times 0.185^2) - \exp(0.02) = 3.86\%$ level equity premium[9]), a retirement income benefit replacement rate of 40%, and the labor income risk of a college graduate (standard deviation of log permanent income shocks of 0.130, log temporary income shocks of 0.242). Expected annual real labor income is $100,000 through age 66 (the last year of working life) and $40,000 thereafter.

Expected age-56 labor income is discounted by dividing it by gross discount rate $1 + r_{y,55}$, where $r_{y,55}$ equals the dot product of the parameter values and their corresponding coefficients in the third column of Table 1: $0.087 \times 7/10 - 0.267 \times 0.02 + 1.132 \times 0.02 + 4.332 \times 0.130^2 + 0.028 \times 0.242^2 + 0.010 \times 0.40 - 0.149 \times 55/100 + 0.142 \times (55/100)^2 - 0.020 = 0.0981$.

Expected age-57 labor income is discounted by dividing it by the cumulative gross discount rate $(1 + r_{y,55})(1 + r_{y,56})$, where $r_{y,56} = 0.087 \times 7/10 - 0.267 \times 0.02 + 1.132 \times 0.02 + 4.332 \times 0.130^2 + 0.028 \times 0.242^2 + 0.010 \times 0.40 - 0.149 \times 56/100 + 0.142 \times (56/100)^2 - 0.020 = 0.0981$.

Expected age-67 labor income is discounted by dividing it by $(1 + r_{y,55})(1 + r_{y,56}) \cdots (1 + r_{y,66})$, where we compute $r_{y,66}$ using the coefficients in the third column of Table 2 because age-67 income comes from the retirement benefit: $0.0003 \times 7/10 - 0.217 \times 0.02 + 0.893 \times 0.02 + 0.476 \times 66/100 - 0.295 \times (66/100)^2 - 0.166 = 0.0334$.

The third column of Table 3 contains the one-period-ahead gross discount rates $1 + r_{y,t-1}$ applied at each age $t - 1$ to income arriving at t. The fourth column contains the cumulative gross discount rates used by the 55-year-old to discount each future year's income, which is the product of the one-period-ahead gross discount rates from age 55 up to the year before the given future year. The final column lists the discounted value of income at each age, the sum of which yields

[9] The expectation of a lognormally distributed random variable whose log has an expectation of μ and a variance of σ^2 is $\exp(\mu + 0.5\sigma^2)$.

human capital value H of \$924,805.

Suppose this individual is allocating a financial portfolio worth \$1,000,000. In the absence of human capital, her optimal equity share would be given by equation (8) as 15.5%. Adjusting for human capital entails multiplying this percentage by $1 + H/W = 1 + 924{,}805/1{,}000{,}000 = 1.92$, giving an optimal equity share of 30%. As a comparison, when we calculate the actual optimal equity share for this individual using dynamic programming, it is 33%. Note that this expected earnings trajectory is not one that was used to form our discount rate approximations, so this exercise is an out-of-sample test of our approximation's accuracy.

If alternatively, we had chosen to use discount rates that do not vary by age except across the retirement threshold, the one-period ahead discount rate would be 0.102 for all working life years and 0.0367 during all retirement years. The resulting human capital value is \$907,681, which also yields an optimal equity share of 30%.

4 Approximation Accuracy

We systematically assess our approximation's accuracy in three ways.

First, we regress each optimal equity share on its corresponding approximately optimal equity share computed using our procedure. Each point in our (age, cash-on-hand as a multiple of permanent income) state space grid for every parameter set we consider constitutes an (approximately optimal equity share, optimal equity share) observation pair—80 million observations in total.[10] This regression yields a slope coefficient of 0.946, an intercept of 0.054, and an R^2 of 0.990.[11] Figure 6 shows a bin-scatter plot of average optimal equity share within 1

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[10] We have 5,103 parameter sets, 78 ages, and 201 cash-on-hand grid points, yielding 5,103 × 78 × 201 = 80,004,834 observations. Because some parameters are relevant during working life but irrelevant in retirement, for each parameter set, there are 80 other parameter sets that have identical solutions during retirement. To avoid overweighting working life relative to retirement, we do not exclude the duplicate observations from the regression.

[11] One reason the R^2 is higher for these equity allocation regressions than for the discount rate regressions in Tables 1 and 2 is that in equation (9), the optimal equity share is less sensitive to discount rate movements of magnitudes comparable to our discount rate approximation errors than to financial wealth movements of the magnitudes we consider.

percentage point wide approximately optimal equity share bins. Our procedure yields an equity share that is somewhat too aggressive when the optimal equity share is below 83% and slightly too conservative when the optimal equity share is above 83%.[12]

Second, we compute the root mean square difference between the equity portfolio share recommended by our approximation and the actual optimum. Across all the parameter sets and state space grid points, the root mean square error is 3.66 percentage points. Figure 7 shows the distribution of the root mean square error across parameter sets, where the height of the bars indicates the number of parameter sets. Sixty-six percent of parameter sets have a root mean square error of 4 percentage points or less, and 95% have a root mean squared error of 5 percentage points or less.

Third, we compute the utility cost of following our approximate rule. A shortcoming of the previous two evaluation criteria is that they equally weight points in the state space that are likely and unlikely to be reached, and recommendation errors that have small and large welfare consequences. We consider a 22-year-old individual whose cash-on-hand equals current labor income. Allowing the individual to optimize her consumption policy given the asset allocation rule followed, we compute expected discounted lifetime utility under the optimal asset allocation rule and under our approximate rule. Each of these discounted utility values corresponds to the expected discounted utility delivered by a constant level of consumption while the agent is alive. The difference between these two levels is our measure of welfare loss. We calculate the average of this welfare loss across our 5,103 parameter sets.

Table 4 shows that across all parameter sets, the average welfare loss from using our rule instead of the optimal rule is equivalent to a 0.06% reduction in lifetime consumption. Welfare losses are similar across different levels of risk aversion, equity premium, and education. The other columns show that welfare losses under other possible portfolio rules are significantly larger than under our rule. A perennial 100% equity allocation results in the largest average welfare loss—a

[12] One could obtain improved asset allocation recommendations by multiplying our procedure's equity share by 0.946 and adding 0.054. We do not add this extra step because the added complexity would yield only small welfare improvements.

11.85% reduction in lifetime consumption, driven by enormous losses for individuals with high risk aversion. Next most costly is permanent equity non-participation, which is equivalent to a 7.86% reduction in lifetime consumption. Perhaps not coincidentally, neither of these investment strategies is commonly recommended. Two popular rules of thumb yield modest welfare losses, although neither do as well as our approximate rule. A lifetime of 60% equities reduces welfare by 3.75%, while investing 100 minus your age percent in stocks reduces welfare by 2.00%.

5 Extension: Positive Correlation Between Labor Income Shocks and Stock Returns

In this section, we develop approximately optimal asset allocations when labor income shocks are positively correlated with stock market returns. Although the contemporaneous correlation between labor income shocks and stock returns is nearly zero in aggregate, when Campbell et al. (2001) regress log labor income shocks in 27 industry × education level subsamples on excess stock returns lagged one year, they estimate regression coefficients of between -0.18 and 0.32. The vast majority of these lagged coefficients are not statistically distinguishable from zero, but for the aggregate population, the lagged coefficient is 0.08 and statistically significant. Campbell et al. (2001) treat these positive lagged correlations as if they were contemporaneous correlations when applying the CGM model.

Viceira (1998) and Campbell and Viceira (2002) derive an analytic approximation to the optimal equity share in a one-period setting where nontradable labor income is imperfectly correlated with the risky asset return. Their formula inspires the conjecture that the following approximation would work well in our setting:

$$\hat{\alpha}_t = \max\{0, \min\{1, \alpha^*(1 + h_t/w_t) - \left(\beta_{perm} + \phi\beta_{temp}\right) h_t/w_t\}\} \tag{19}$$

where $\beta_{perm} = cov(u_t, r_t)/\sigma_r^2$ is the population coefficient from regressing the log permanent income shock on the log stock market return, $\beta_{temp} = cov(\epsilon_t, r_t)/\sigma_r^2$ is the population coefficient from regressing the log temporary income shock on the log stock market return, and ϕ is a constant to be estimated. Intuitively, a fraction $\beta_{perm} + \phi\beta_{temp}$ of human capital is an implicit equity investment, so the optimal financial equity investment is reduced by $(\beta_{perm} + \phi\beta_{temp})H_t$ dollars, or $(\beta_{perm} + \phi\beta_{temp})h_t$ times permanent income, relative to when labor income shocks are uncorrelated with stock market returns. We also conjecture that calculating h_t in equation (19) using the same approximate discount rates as in the uncorrelated human capital case will yield good results.

We first consider positively correlated permanent income shocks. Figure 8 plots optimal equity shares for a particular parameter set when β_{perm} = 0, 0.1, 0.2, or 0.3, and β_{temp} = 0. As expected, the more positively correlated labor income is with stock returns, the more conservative is the optimal financial portfolio allocation. When $\beta_{perm} = 0.3$, the optimal equity share increases in invested wealth. This phenomenon arises in equation (19) when the Merton solution α^* (which is 0.271 in this case) is smaller than β_{perm}. In such a scenario, the implicit equity exposure in human capital is so high that at low levels of financial wealth, the individual's stock market exposure is above her ideal even when there is no stock in her financial portfolio. It is only when financial wealth rises sufficiently that the individual needs to buy some stock to keep her stock market exposure at its ideal level.

Also surprising is the fact that the downward shift in equity share is nearly as large at age 65 as it is at age 30. This is an artifact created by the retirement income benefit being determined entirely by permanent income at age 66 in the CGM model. A negative stock market return at age 66 that percolates into a -1% permanent income shock that year forces retirement income to be permanently 1% lower, so a 65-year-old has substantial stock market risk embedded in human capital despite having only one working year remaining. In contrast, in the U.S. Social Security system, a low income realization at age 65 has only a small effect on one's Social Security benefit because the benefit payment is determined by the average of 35 years of indexed labor income.

We use equation (19) for equity shares through age 65 and equation (9) for equity shares from age 66 onwards.[13] We consider cases where β_{perm} = 0.1, 0.2, or 0.3, and $\beta_{temp} = 0$, for all 5,103 parameter sets. The root mean square approximation error for equity share before age 66 across these 3 × 5,103 = 15,309 scenarios is 3.67 percentage points.[14] Regressing optimal equity share on approximately optimal equity share before age 66 yields a slope coefficient of 0.988, an intercept of 0.0004, and an R^2 of 0.990. The average welfare loss from using equations (9) and (19) instead of the optimal rule is equivalent to only a 0.02% drop in lifetime consumption.

We next study the case where temporary income shocks are positively correlated with stock market returns. Figure 9 contains optimal equity share graphs when $\beta_{perm} = 0$ and β_{temp} = 0, 0.1, 0.2, or 0.3. Although a positive correlation between temporary labor income shocks and the stock market also has a negative effect on equity shares, the magnitude of this effect is negligible.

We estimate the value of ϕ that minimizes the mean squared deviation between the output of (19) and the actual optimal equity allocations at all state space grid points and parameter value sets for β_{temp} = 0.1, 0.2, or 0.3, and $\beta_{perm} = 0$.[15] We obtain $\phi = 0.0045$, implying that positively correlated temporary income shocks barely matter for the optimal equity share. Therefore, the formula $\max\{0, \min\{1, \alpha^*(1 + H/W) - \beta_{perm} H/W\}\}$ is adequate in practice for calculating the optimal equity share during working life. The root mean square error before age 66 is 5.47 percentage points whether or not $\phi = 0$ or 0.0045.[16] With $\phi = 0.0045$, regressing

[13] If we allow a constant ξ to multiply β_{perm} in equation (19) and choose its value to minimize the root mean square approximation error, we obtain $\xi = 1.014$, which is essentially identical to the value of 1 we use. The root mean square error if $\xi = 1.014$ is 3.52%.

[14] For all root mean square error calculations and regressions of optimal equity share on approximately optimal equity share in this section, we exclude cases where the individual endogenously chooses to save less than 0.1% of current permanent income. The welfare consequences of asset allocation choices are trivial for such a small amount of money, making the numerical solutions from the dynamic programming code unstable. Figures 8 and 9 also exclude these cases.

[15] We exclude cases where the individual saves less than 0.1% of current permanent income in estimating ϕ.

[16] The root mean square error is larger when temporary income shocks are correlated with stock returns than when permanent income shocks are correlated with stock returns because in the former case, there are more instances where the optimal equity share has a non-monotonic relationship with wealth—falling for small values of wealth, quickly rising back to 100% as wealth increases modestly, and then resuming its usual weakly decreasing pattern. Such cases are most common at young ages and around retirement.

optimal equity share on approximately optimal equity share before age 66 yields a slope coefficient of 0.935, an intercept of 0.059, and an R^2 of 0.973. The average welfare loss from using the approximation is 0.042% if $\phi = 0.0045$ and 0.043% if $\phi = 0$.

6 Conclusion

We have developed a methodology for obtaining lifecycle asset allocation advice that closely approximates the recommendations from the canonical Cocco, Gomes, and Maenhout (CGM 2005) model. The methodology is economically interpretable, mathematically straightforward, and easily implemented. Unlike pre-existing analytic formulas, our recommendations are derived within a reasonably realistic setting. Unlike pre-existing rules of thumb, our recommendations are tailored to individual preferences and circumstances.

Real life is more complicated than the CGM setting, so the CGM solutions are probably not exactly optimal for a given individual. Deriving solutions for more realistic settings is an obvious direction for future work. However, as statistician George Box famously wrote, "All models are wrong, but some are useful." A constraint on future research in practical finance is that in order to be useful, solutions for more complex, realistic settings should remain easily computable by a layperson and their economic rationale transparent enough to be persuasive.

Appendix

Numerical Solution Methodology

We solve the model numerically using backward induction. At each age, we discretize the normalized cash-on-hand state variable into 201 points from 0.25 to 50, which are exponentially spaced with more density near zero (each value is 1.0268 times the prior value). We use exponentially spacing because the value function tends to be more concave at low levels of cash-on-hand and close to linear at high levels. Exponential spacing allocates more points where the function is more curved, improving the accuracy of the solution.

We use grid search over choice variables to find the optimal decisions at each age × cash-on-hand grid point, considering 101 equally spaced points for equity shares between 0% and 100%, and 101 equally spaced points for consumption as a multiple of permanent income. We use CGM's algorithm for determining the range of consumption levels over which search occurs. Let i index cash-on-hand values on the grid. At each age, for $i = 1$ (the lowest cash-on-hand grid point), the consumption range we search over is 25% to 99.9% of cash-on-hand. For $i = 2$ to 9, the marginal propensity to consume (MPC) as we move from $i - 1$ to i is constrained to be between 0 and 1. For $i \geq 10$, the MPC is constrained to be between 0 and the average MPC from $i - 9$ to $i - 1$.

Normally distributed random variables are approximated as discrete random variables using a nine-point Gaussian quadrature following Tauchen and Hussey (1991). Following the method of Carroll (2022), we transform the CRRA value function into a quasi-linear form that is equivalent to the Epstein and Zin (1989) value function, which allows us to accurately evaluate the value function at points not on the state space grid points using linear interpolation and extrapolation. Following CGM, when evaluating the value function at cash-on-hand values below 0.25, we set its value equal to its value when cash-on-hand equals 0.25, which prevents the extrapolation procedure from assigning a negative value to the value function.

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Table 1. Pre-retirement labor income discount rate approximation regressions

This table shows coefficients from regressions where the dependent variable is the one-period-ahead labor income discount rate at each age from 22 to 65 for each of the 5,103 sets of parameter values we consider. In the first column, the explanatory variables are relative risk aversion, log equity premium, log risk-free rate, permanent income shock variance, transitory income shock variance, and retirement income replacement rate. The second column adds age as an explanatory variable, and the third column adds the square of age. The fourth column adds interactions of each of the non-age variables with age and the square of age as explanatory variables; the coefficients on these interactions are not shown. Standard errors are in parentheses below the point estimates.

Relative risk aversion ÷ 10	0.087 (0.000)	0.087 (0.000)	0.087 (0.000)	0.055 (0.001)
Log equity premium	-0.267 (0.002)	-0.267 (0.002)	-0.267 (0.002)	0.430 (0.025)
Log risk-free rate	1.132 (0.002)	1.131 (0.002)	1.132 (0.002)	1.225 (0.025)
Permanent income shock variance	4.332 (0.007)	4.332 (0.006)	4.332 (0.006)	5.888 (0.069)
Transitory income shock variance	0.028 (0.001)	0.028 (0.001)	0.028 (0.001)	0.015 (0.010)
Retirement income replacement rate	0.010 (0.000)	0.010 (0.000)	0.010 (0.000)	0.124 (0.001)
Age ÷ 100		-0.026 (0.000)	-0.149 (0.001)	0.464 (0.009)
$(\text{Age} \div 100)^2$			0.142 (0.001)	-0.737 (0.010)
Constant	-0.055 (0.000)	-0.044 (0.000)	-0.020 (0.000)	-0.106 (0.002)
Age interactions	No	No	No	Yes
Observations	224,532	224,532	224,532	224,532
Adj. R-squared	0.864	0.880	0.886	0.907

Table 2. Retirement labor income discount rate approximation regressions

This table shows coefficients from regressions where the dependent variable is the one-period-ahead labor income discount rate at each age from 66 to 99 for each of the 63 sets of parameter values we consider for the retirement period. In the first column, the explanatory variables are relative risk aversion, log equity premium, and log risk-free rate. The second column adds age as an explanatory variable, and the third column adds the square of age. The fourth column adds interactions of each of the non-age variables with age and the square of age as explanatory variables; the coefficients on these interactions are not shown. Standard errors are in parentheses below the point estimates.

Relative risk aversion	0.0003	0.0003	0.0003	0.056
÷ 10	(0.001)	(0.000)	(0.000)	(0.030)
Log equity premium	-0.217	-0.217	-0.217	8.447
	(0.012)	(0.012)	(0.010)	(0.733)
Log risk-free rate	0.893	0.893	0.893	2.666
	(0.012)	(0.012)	(0.010)	(0.733)
Age ÷ 100		-0.011	0.476	1.256
		(0.001)	(0.016)	(0.078)
$(\text{Age} \div 100)^2$			-0.295	-0.769
			(0.009)	(0.047)
Constant	0.023	0.032	-0.166	-0.482
	(0.001)	(0.001)	(0.006)	(0.032)
Age interactions	No	No	No	Yes
Observations	2,142	2,142	2,142	2,142
Adj. R-squared	0.721	0.736	0.819	0.841

Table 3. Human capital value calculation example

This table contains a human capital value calculation for a 55-year-old with relative risk aversion of 7, a log real risk-free rate of 2%, a log equity premium of 2%, a retirement income benefit replacement rate of 40%, and the labor income risk of a college graduate. The first column contains the age t at which the expected real labor income (conditional on being alive) in the second column arrives. The third column shows the one-period-ahead gross discount rate that is applied at age $t-1$ to income arriving at t. The fourth column shows the cumulative gross discount rate that the 55-year-old applies to income arriving at each future age, which is the product of all one-period-ahead gross discount rates from age 55 up to the year before that future age. The final column is each year's discounted expected income from the perspective of the 55-year-old, which is expected income divided by the cumulative gross discount rate. In the bottom row, the value of human capital is computed as the sum of discounted expected income through age 100.

Age t	Expected income	Gross discount rate $1 + r_{y,t-1}$	Cumulative gross discount rate	Discounted expected income
56	100,000	1.0981	1.0981	91,070
57	100,000	1.0981	1.2058	82,931
58	100,000	1.0983	1.3243	75,512
59	100,000	1.0984	1.4546	68,747
60	100,000	1.0986	1.5980	62,579
61	100,000	1.0988	1.7558	56,953
62	100,000	1.0990	1.9297	51,823
63	100,000	1.0993	2.1212	47,144
64	100,000	1.0995	2.3323	42,876
65	100,000	1.0999	2.5652	38,983
66	100,000	1.1002	2.8222	35,433
67	40,000	1.0334	2.9165	13,715
68	40,000	1.0342	3.0163	13,261
69	40,000	1.0350	3.1219	12,813
70	40,000	1.0357	3.2334	12,371
71	40,000	1.0364	3.3510	11,937
72	40,000	1.0370	3.4749	11,511
73	40,000	1.0375	3.6053	11,095
74	40,000	1.0380	3.7423	10,689
75	40,000	1.0384	3.8861	10,293
76	40,000	1.0388	4.0369	9,909
77	40,000	1.0391	4.1947	9,536
78	40,000	1.0393	4.3598	9,175
79	40,000	1.0395	4.5321	8,826
80	40,000	1.0397	4.7119	8,489
81	40,000	1.0397	4.8991	8,165
82	40,000	1.0397	5.0938	7,853
83	40,000	1.0397	5.2959	7,553
84	40,000	1.0396	5.5056	7,265
85	40,000	1.0394	5.7226	6,990
86	40,000	1.0392	5.9469	6,726

87	40,000	1.0389	6.1783	6,474
88	40,000	1.0386	6.4165	6,234
89	40,000	1.0382	6.6614	6,005
90	40,000	1.0377	6.9125	5,787
91	40,000	1.0372	7.1695	5,579
92	40,000	1.0366	7.4319	5,382
93	40,000	1.0360	7.6992	5,195
94	40,000	1.0353	7.9707	5,018
95	40,000	1.0345	8.2458	4,851
96	40,000	1.0337	8.5236	4,693
97	40,000	1.0328	8.8033	4,544
98	40,000	1.0319	9.0840	4,403
99	40,000	1.0309	9.3646	4,271
100	40,000	1.0298	9.6441	4,148
			Human capital *H*:	**924,805**

Table 4. Welfare losses from following various portfolio rules

This table shows the average welfare loss across the parameter sets indicated in the row label from using the portfolio allocation rule in the column label instead of the optimal rule, stated in terms of the reduction in level lifetime consumption that would generate the same discounted expected utility loss. The welfare losses are from the perspective of a 22-year-old whose cash-on-hand equals this period's labor income.

	Our rule	0% equity	60% equity	100% equity	(100 – age)% equity
All parameter sets	0.06%	7.86%	3.75%	11.75%	2.00%
RRA = 4	0.03%	7.93%	1.58%	0.56%	2.11%
RRA = 10	0.07%	7.09%	9.27%	29.55%	4.11%
Log equity premium = 2%	0.05%	5.24%	4.72%	14.78%	2.21%
Log equity premium = 4%	0.06%	10.54%	2.95%	8.85%	1.93%
College graduate	0.04%	5.55%	2.76%	9.21%	1.32%
High school graduate	0.06%	8.63%	3.96%	12.08%	2.17%
No high school	0.07%	9.40%	4.54%	13.94%	2.51%

Figure 1. Optimal equity shares for one parameter set

This figure shows the optimal fraction of the financial portfolio allocated to the risky asset as a function of $w_t \equiv x_t - c_t$ (cash-on-hand at the beginning of the period minus current consumption normalized by current permanent income) at various ages for an investor with relative risk aversion of 7 facing the labor income process of a college graduate, a log risk-free rate of 2%, a log equity premium of 2%, and a retirement income benefit replacement rate of 40%.

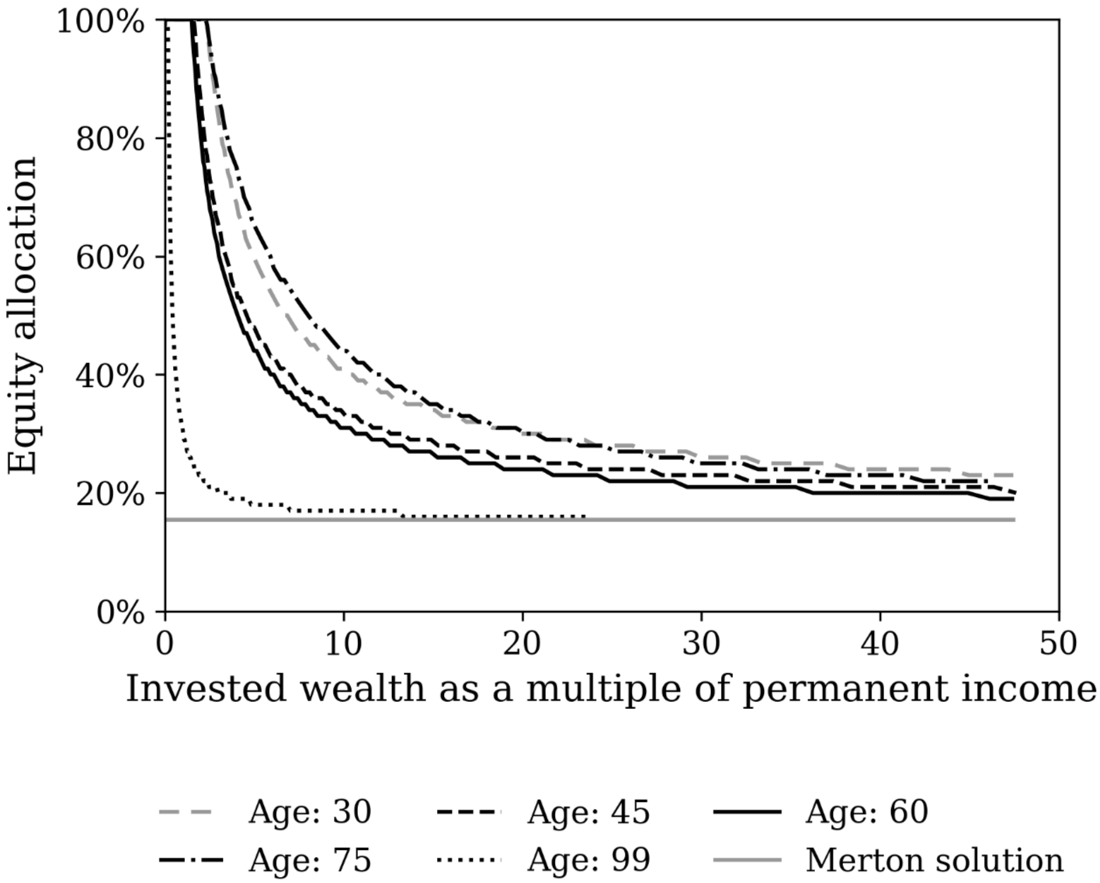


Figure 2. Labor income discount rates under three different risk aversion values

This figure shows one-period-ahead discount rates for labor income by age for an individual with a coefficient of relative risk aversion equal to 4, 7, or 10. Under all scenarios, the individual faces the labor income process of a college graduate, a log risk-free rate of 2%, a log equity premium of 2%, and a retirement income benefit replacement rate of 40%.

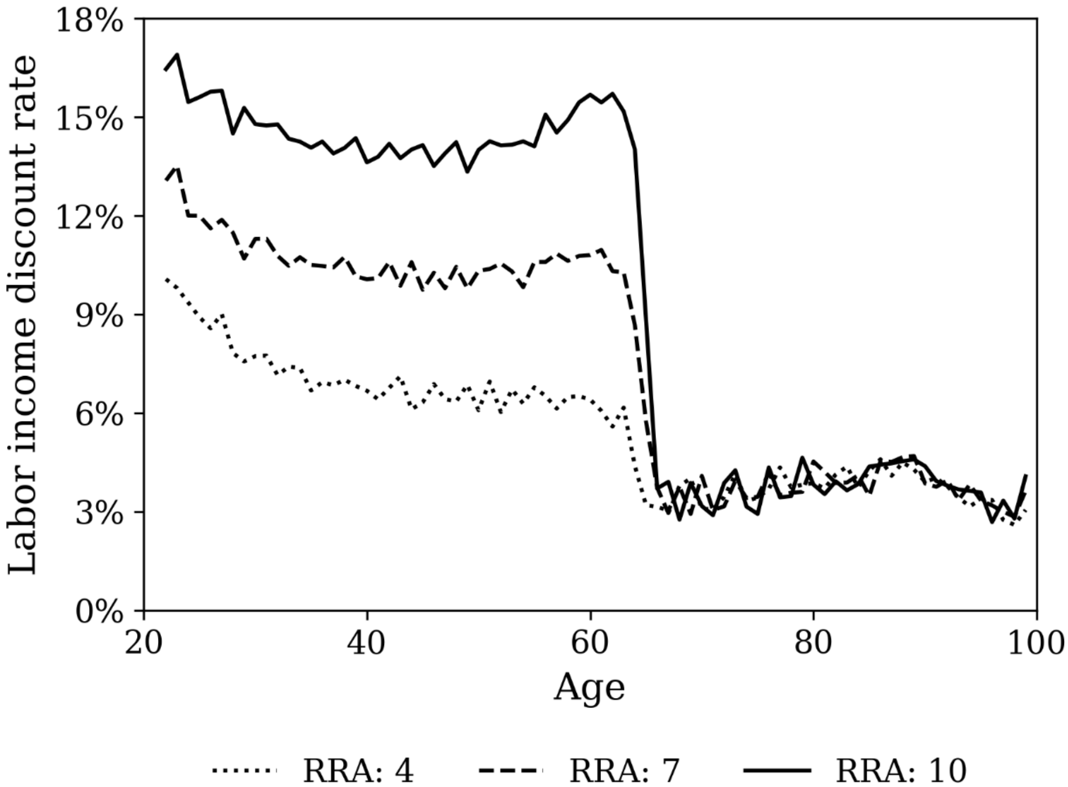


Figure 3. Labor income discount rates under three different retirement income replacement rates

This figure shows one-period-ahead discount rates for labor income by age for an individual facing a 40%, 60%, or 80% retirement income benefit replacement rate with relative risk aversion of 4, 7, or 10. Under all scenarios, the individual faces the labor income process of a college graduate, a log equity premium of 2%, and a log risk-free rate of 2%.

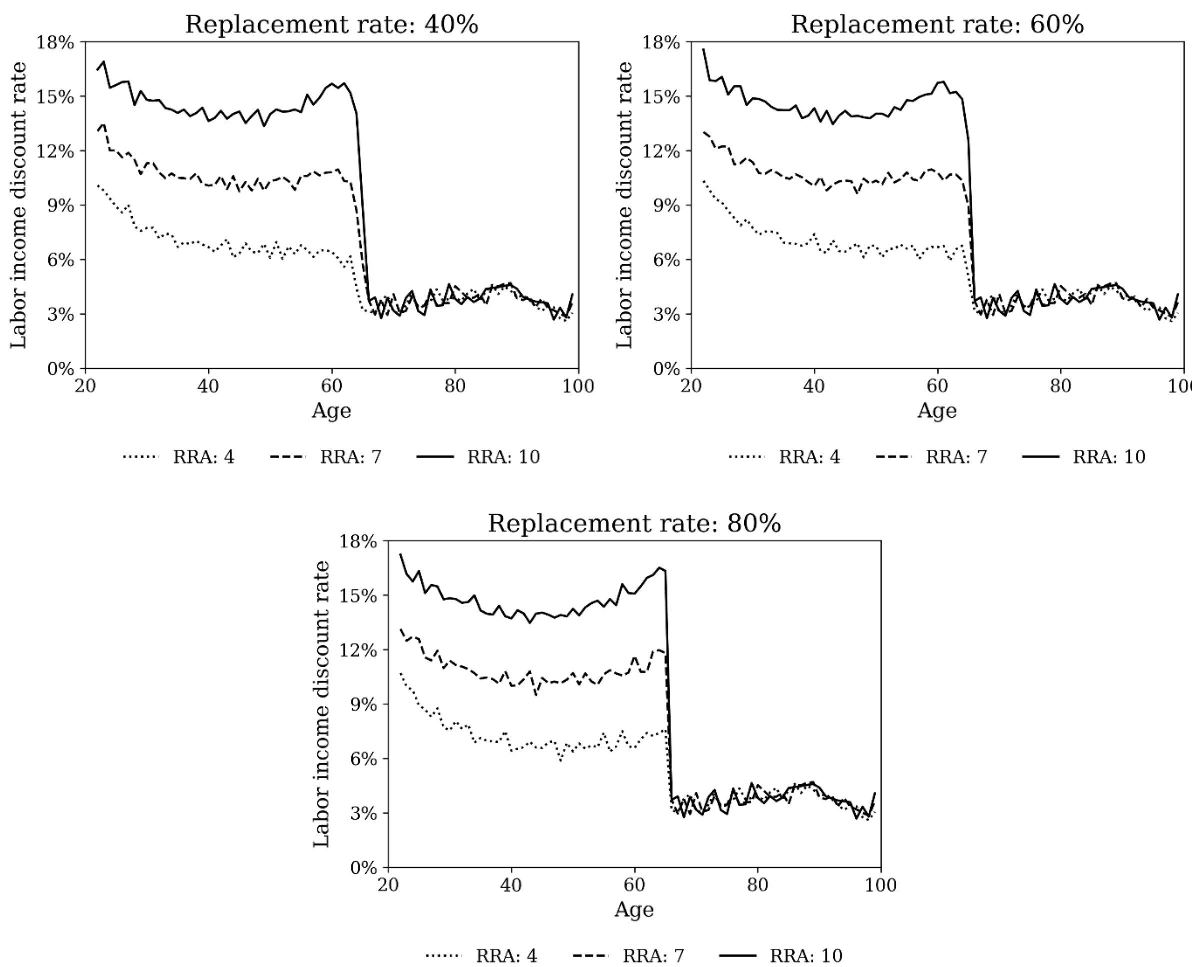


Figure 4. Labor income discount rates under three different risk-free rates

This figure shows one-period-ahead discount rates for labor income by age for an individual facing a 0%, 1%, or 2% log risk-free interest rate. Under all scenarios, the individual has relative risk aversion of 7 and faces the labor income process of a college graduate, a log equity premium of 2%, and a retirement income benefit replacement rate of 40%.

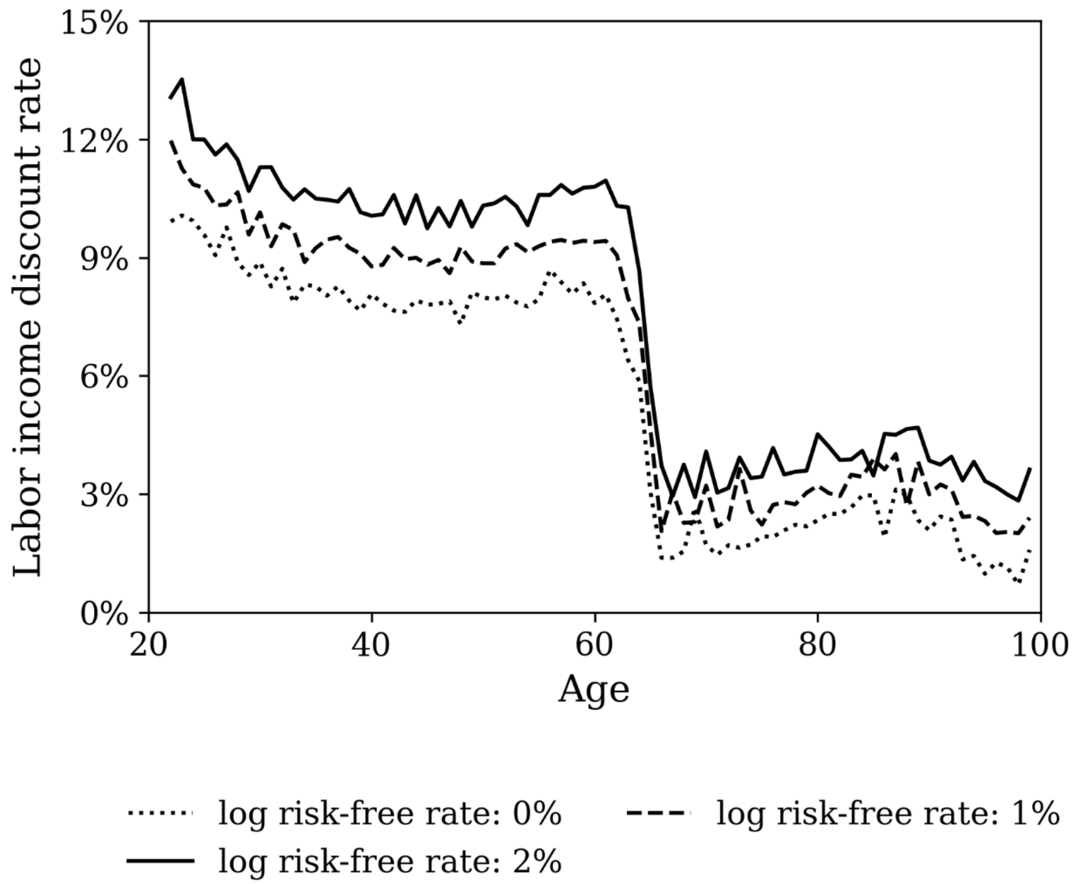


Figure 5. Labor income discount rates under three different equity premia

This figure shows one-period-ahead discount rates for labor income by age for an individual facing a 2%, 3%, or 4% log equity premium. Under all scenarios, the individual has relative risk aversion of 7 and faces the labor income process of a college graduate, a log risk-free rate of 2%, and a retirement income benefit replacement rate of 40%.

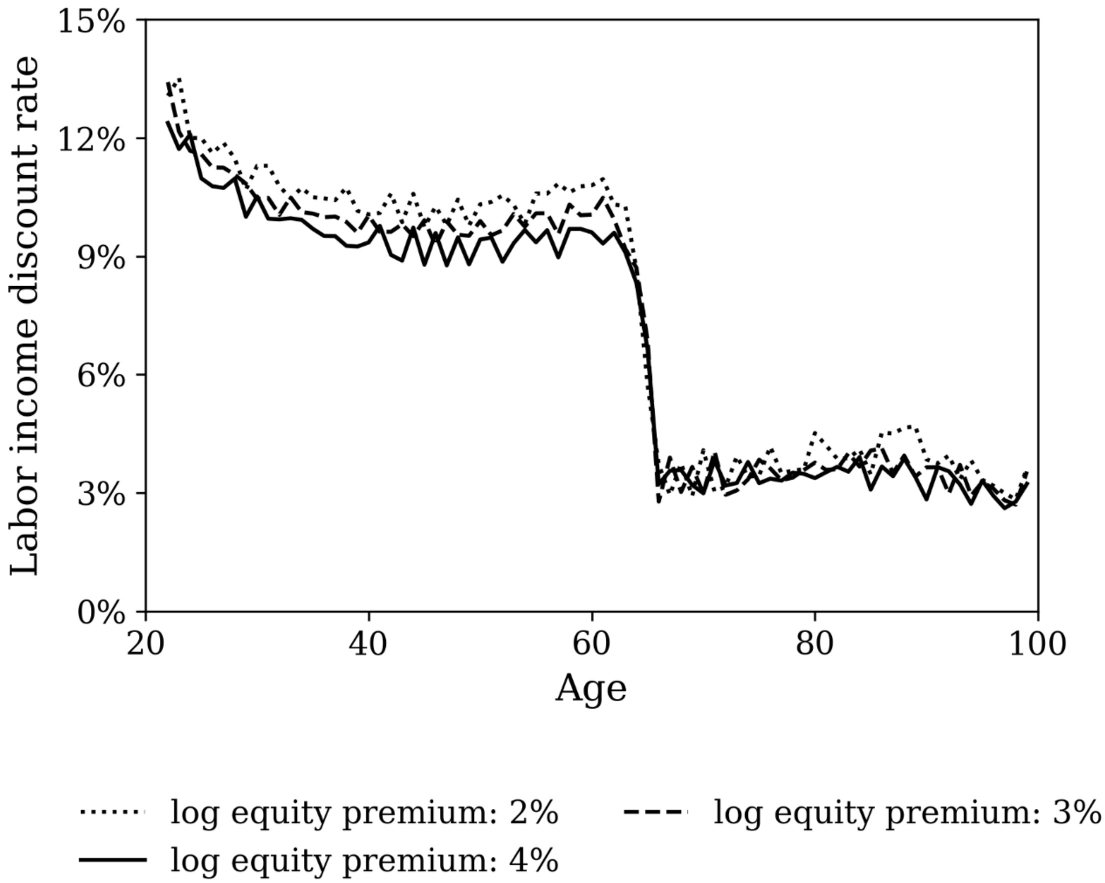


Figure 6. Optimal equity share versus approximately optimal equity share

This bin-scatter graph plots the optimal equity share against the corresponding approximately optimal equity share, computed using the procedure developed in this paper. Each point represents a bin of observations grouped by the approximately optimal equity share. The observations used to construct this graph are the (approximately optimal equity share, optimal equity share) pairs at every (age, cash-on-hand as a multiple of permanent income) point in our state space grid for every parameter set we consider. The dashed line is the 45-degree line.

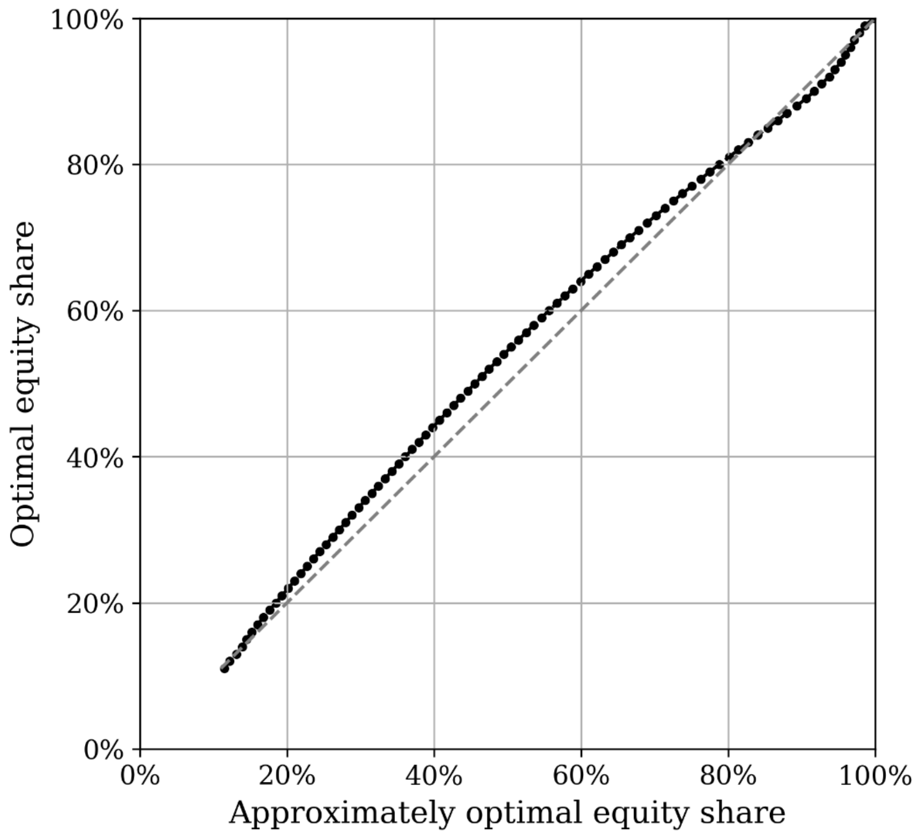


Figure 7. Distribution of root mean square approximation error across parameter sets

This figure shows the number of parameter sets we consider for which the square root of the mean squared deviation between our approximately optimal equity share and the actual equity share, computed over all our state space grid points, equals the value on the horizontal axis.

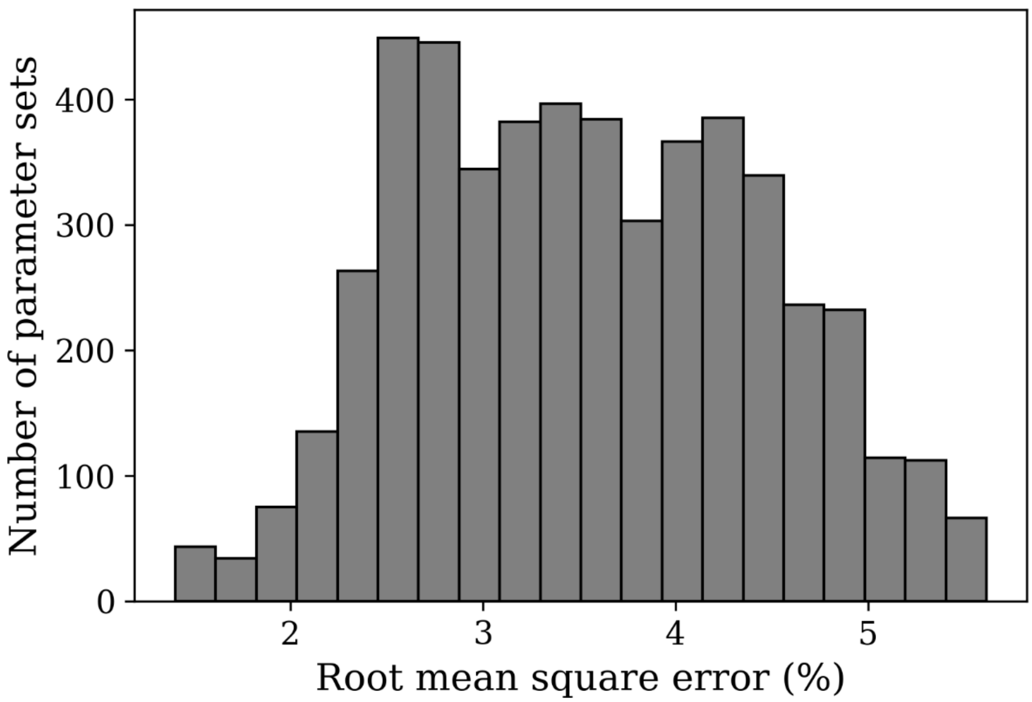
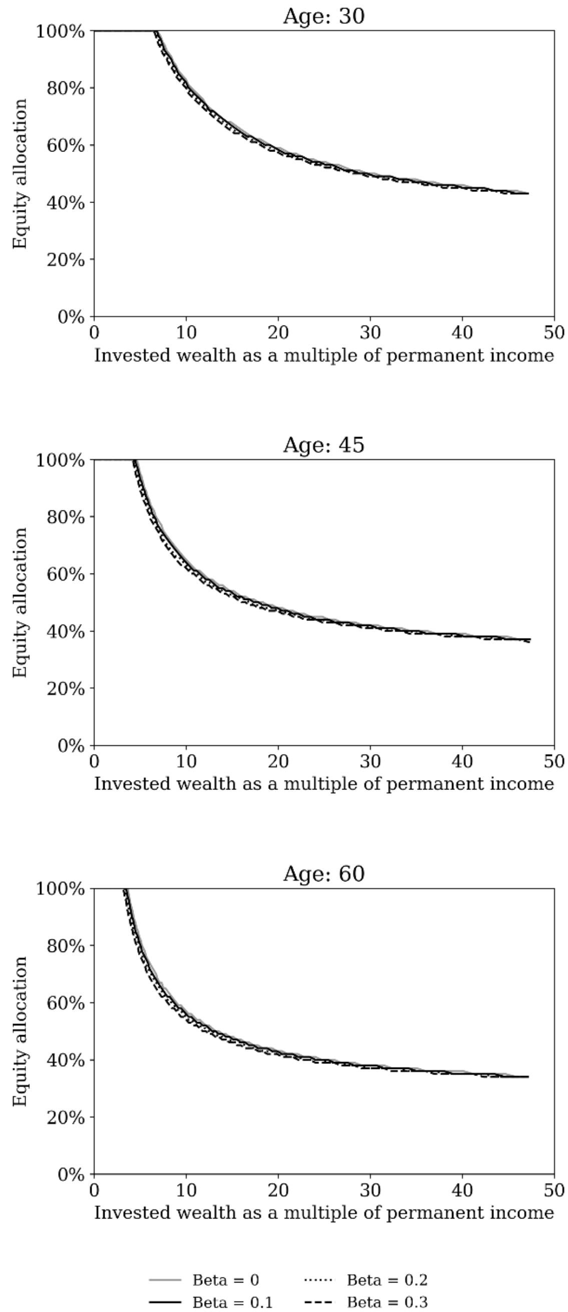


Figure 8. Optimal equity allocations when permanent labor income shocks are positively correlated with stock market returns

This figure shows the optimal fraction of the financial portfolio allocated to the risky asset as a function of $w_t \equiv x_t - c_t$ (cash-on-hand at the beginning of the period minus current consumption normalized by current permanent income) at various ages for an investor with relative risk aversion of 4 facing the labor income process of a college graduate, a log risk-free rate of 2%, a log equity premium of 2%, and a retirement income benefit replacement rate of 40%. The temporary labor income shock is uncorrelated with stock returns, and the log permanent labor income shock has a coefficient when regressed on log stock returns of 0, 0.1, 0.2, or 0.3.

Figure 9. Optimal equity allocations when temporary labor income shocks are positively correlated with stock market returns

This figure shows the optimal fraction of the financial portfolio allocated to the risky asset as a function of $w_t \equiv x_t - c_t$ (cash-on-hand at the beginning of the period minus current consumption normalized by current permanent income) at various ages for an investor with relative risk aversion of 4 facing the labor income process of a college graduate, a log risk-free rate of 2%, a log equity premium of 2%, and a retirement income benefit replacement rate of 40%. The permanent labor income shock is uncorrelated with stock returns, and the log temporary labor income shock has a coefficient when regressed on log stock returns of 0, 0.1, 0.2, or 0.3.



Online Appendix Figure 1. Labor income discount rates when deterministic portion of labor income is constant during working life

This figure shows one-period-ahead discount rates for labor income by age for an individual with a coefficient of relative risk aversion equal to 4, 7, or 10. The gray lines correspond to scenarios where the individual faces log labor income whose deterministic portion is constant during working life and whose risk is that of a college graduate, a log risk-free rate of 2%, a log equity premium of 2%, and a retirement income benefit replacement rate of 40%. The black lines correspond to scenarios that are the same as for the gray lines, except that the deterministic portion of labor income during working life follows the path estimated for college graduates.