Notes on Renewal Processes

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Abstract

This is a set of notes on renewal processes that formed the basis for a lecture in the PhD class Modeling Operational Processes offered in the Spring of 2025 at the Yale School of Management.

1. Preliminaries

Let X be a non-negative continuous random variable with cumulative distribution function (cdf) $F_X(t) = \Pr\{X \leq t\}$, probability density function (pdf) $f_X(t) = \frac{d}{dt}F_X(t)$, finite mean $E(X) = \tau$ and variance $Var(X) = \sigma^2$. Starting from time 0, "arrivals" or "events" or "*renewals*" occur with *interarrival times* independently and identically distributed (iid) as random variable X. Let T_n denote the time (measured from the start of the process at time 0) of the n^{th} arrival. Clearly

$$T_n = \sum_{i=1}^{n} X_i$$

where all of the X_i's are iid as random variable X.

The cdf and pdf of random variable T_n are notated by $F_{T_n}(t) = \Pr\{T_n \leq t\}$ and $f_{T_n}(t) = \frac{d}{dt}F_{T_n}(t)$. Note the convolution relations

$$f_{T_{n+1}}(t) = \int_0^t f_{T_n}(t-s) f_X(s) ds$$

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and

$$F_{T_{n+1}}(t) = \int_0^t F_{T_n}(t-s) f_X(s) ds.$$

Roughly speaking, to find the probability (density) that the $n+1^{st}$ renewal occurs at time t, we note that if the first renewal occurs at time s, then an additional n renewals must occur over the remaining $t-s$ time periods; the integral accounts for all possible first renewal times s between 0 and t. Similarly, for the $n+1^{st}$ renewal to occur at or before time t, again note that if the first renewal occurs at time s, then the remaining n renewals must occur within the remaining $t-s$ time available.

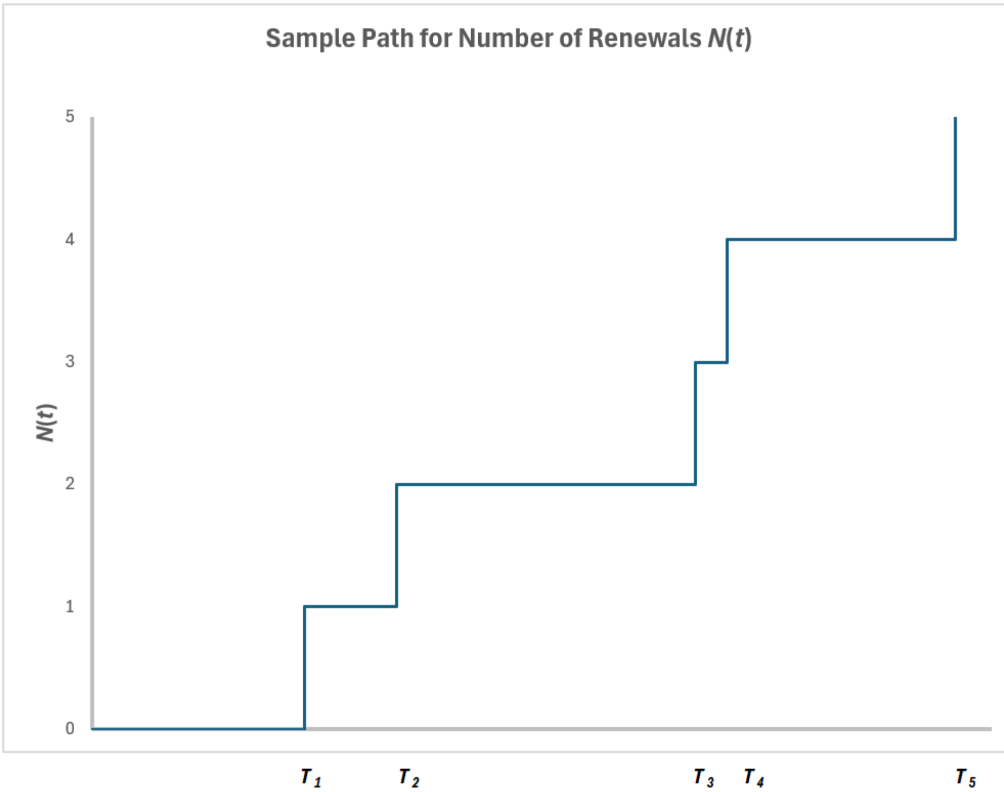
2. Introducing $N(t)$, the Number of Renewals by Time t

It is easy to study the timing of the n^{th} renewal by considering the probability laws of random variable T_n. Now we will turn the problem around, and consider the *number of renewals* that occur by time t (where by convention time 0 is taken as the start of the process). We denote the number of renewals that occur by time t as $N(t)$, and define this variable mathematically as

$$N(t) = \max\{n | T_n \leq t\}, \ t \geq 0.$$

Fix time to t, and ask yourself what is the largest value of n such that $T_n \leq t$? The answer to that question must be $N(t)$, the number of renewals by time t. A sample path showing the evolution of $N(t)$ over time is shown in the figure on the next page.

We would like to understand the random variable $N(t)$, as it is a basic process in operations research. $N(t)$ could correspond to the number of customers who have arrived to a service system by time t, the number of persons who have been infected in a stable disease transmission process by time t, or the number of terror attacks that have been attempted by time t, as examples. We certainly would like to know how to calculate $E(N(t))$, $Var(N(t))$, and if possible $\Pr\{N(t) = n\}$ for $n = 0, 1, 2, ...$ We would also like to be able to deduce asymptotic results describing the behavior of $N(t)$ as t becomes large (that is, as $t \to \infty$). Along the way we will discover how we can do approximate versus exact calculations..



3. Finding the Expected Number of Renewals $E(N(t))$

Suppose a renewal process has just started at time 0, and suppose that the first renewal occurs at some (random) time S, $0 < S \leq t$. The number of renewals that occur by time t must then be given by

$$N(t) = 1 + N(t - S)$$

since the first renewal at time S contributes one, while the remaining number of renewals must occur between time S and t, which amounts to $N(t - S)$ renewals since the process starts over (i.e. renews itself) at time S. Now, S is a random variable corresponding to the time of the first renewal, which means that the probability density of S is exactly the same as the probability density of X, the interarrival time distribution that drives the entire renewal process.

We now seek the expected number of renewals that occur between 0 and t, that is $E(N(t))$. We can obtain a defining equation simply by taking the expected value of $N(t)$ as defined above (where the expectation is with respect to the random variable S). We obtain

$$\begin{aligned} E(N(t)) &= E(1 + N(t - S)) \\ &= \int_0^t ((1 + N(t - s))f_X(s)ds. \end{aligned}$$

The integral only runs from 0 to t as once the value of S exceeds t, the number of renewals that occur within $(0, t]$ equals zero.

The integral equation above is an example of a *renewal equation* (indeed some refer to this as *the* renewal equation). Note that since $\int_0^t f_X(s)ds = \Pr\{T_1 \le t\} = F_{T_1}(t)$ by definition (recall that T_1 is the time of the first renewal), expanding the equation yields

$$E(N(t)) = F_{T_1}(t) + \int_0^t E(N(t - s))f_X(s)ds.$$

We will show how to solve this momentarily, but first we will introduce the *general renewal equation*. Suppose you know the (deterministic) function $g(t)$, and seek to discover the (deterministic) function $z(t)$ that is defined by

$$z(t) = g(t) + \int_0^t z(t - s)f_X(s)ds.$$

This is the general renewal equation. Given $g(t)$ and the interarrival density $f_X(s)$, find the function $z(t)$ defined by the integral equation above. Note that our equation for the expected number of renewals $E(N(t))$ is a special case of this general question with $g(t) = F_{T_1}(t)$, and $z(t) = E(N(t))$.

Returning to the expected number of renewals, we can solve the equation using successive approximations to deduce what the form of the solution must be, and then verify the solution. Let $z(t)$ now refer to $E(N(t))$, and let $z^{(i)}(t)$ denote the i^{th} successive approximation to $z(t)$. The successive approximation scheme proceeds according to the schedule

$$z^{(i+1)}(t) = F_{T_1}(t) + \int_0^t z^{(i)}(t - s)f_X(s)ds.$$

We start with $z^{(0)}(t) = 0$, and inserting this into the approximation schedule yields

$$z^{(1)}(t) = F_{T_1}(t).$$

Now we iterate to obtain

$$\begin{aligned} z^{(2)}(t) &= F_{T_1}(t) + \int_0^t z^{(1)}(t-s) f_X(s) ds \\ &= F_{T_1}(t) + \int_0^t F_{T_1}(t-s) f_X(s) ds \\ &= F_{T_1}(t) + F_{T_2}(t) \end{aligned}$$

where the second term follows from the convolution of the distribution of T_1 with the density of X introduced at the start of these notes. Iterating again we obtain

$$\begin{aligned} z^{(3)}(t) &= F_{T_1}(t) + \int_0^t z^{(2)}(t-s) f_X(s) ds \\ & F_{T_1}(t) + \int_0^t (F_{T_1}(t-s) + F_{T_2}(t-s)) f_X(s) ds \\ &= F_{T_1}(t) + F_{T_2}(t) + F_{T_3}(t) \end{aligned}$$

where we have again taken advantage of the convolution of the distribution of T_n (for $n = 1, 2$) with the density of X. A pattern is clearly emerging which leads to the conjecture that

$$z(t) = \sum_{n=1}^{\infty} F_{T_n}(t).$$

Is this the solution to the renewal equation for $z(t) = E(N(t))$? To see that it is, we verify by writing

$$\begin{aligned} z(t) &= F_{T_1}(t) + \int_0^t z(t-s) f_X(s) ds \\ &= F_{T_1}(t) + \int_0^t (\sum_{n=1}^{\infty} F_{T_n}(t-s)) f_X(s) ds \\ &= F_{T_1}(t) + \sum_{n=2}^{\infty} F_{T_n}(t) \\ &= \sum_{n=1}^{\infty} F_{T_n}(t) \end{aligned}$$

as was to be shown (with the integration result again following from the convolution relating the distribution of T_n (for $n = 1, 2, 3, 4, ...$) and the density of X. We have thus shown that

$$E(N(t)) = \sum_{n=1}^{\infty} F_{T_n}(t).$$

There must be an intuitive explanation for this result, and here it is. Look again at the sample path of the renewal process graphed earlier, and note the following:

The events $N(t) \geq n$ and $T_n \leq t$ are the same events!

If at time t there have been at least n renewals, then the time at which the n^{th} renewal occurred must be less than or equal to t! If $T_n > t$, then it would be impossible for there to have been n renewals by time t since T_n *is* the time of the n^{th} renewal! Working in reverse, if you know that the n^{th} renewal happened at or before time t is reached, then you immediately know that the total number of renewals that occurred by time t must at least equal n. Since the events $N(t) \geq n$ and $T_n \leq t$ are equivalent, it must be true that

$$\Pr\{N(t) \geq n\} = \Pr\{T_n \leq t\}$$

since equivalent events have the same probability. Now, from elementary probability theory, we know that

$$\begin{aligned} E(N(t)) &= \sum_{n=0}^{\infty} n \Pr\{N(t) = n\} \\ &= \sum_{n=0}^{\infty} n(\Pr\{N(t) \geq n\} - \Pr\{N(t) \geq n+1\}) \\ &= \sum_{n=1}^{\infty} \Pr\{N(t) \geq n\} \\ &= \sum_{n=1}^{\infty} \Pr\{T_n \leq t\} \\ &= \sum_{n=1}^{\infty} F_{T_n}(t). \end{aligned}$$

So now we see two different ways to arrive at the same result for expressing $E(N(t))$, the expected number of renewals by time t in a renewal process that starts at time 0.

3.1. Example: The expected number of renewals when the interarrival times are uniformly distributed

Consider a renewal process where the interarrival times X are uniformly distributed between 0 and 1, that is

$$f_X(s) = f_{T_1}(s) = \begin{cases} 1 & 0 \leq s \leq 1 \\ 0 & \text{all other } s \end{cases}$$

It follows that

$$F_{T_1}(t) = \int_0^t f_{T_1}(s)ds = t,\ 0 \leq t \leq 1.$$

Focusing only on values of t that fall between 0 and 1, from the convolutions relating the cumulative distribution of T_n and the density of X we see that

$$\begin{aligned} F_{T_2}(t) &= \int_0^t F_{T_1}(t-s) f_X(s)ds \\ &= \int_0^t (t-s)ds \\ &= \frac{t^2}{2},\ 0 \leq t \leq 1 \end{aligned}$$

and

$$\begin{aligned} F_{T_3}(t) &= \int_0^t F_{T_1}(t-s) f_X(s)ds \\ &= \int_0^t \frac{(t-s)^2}{2} ds \\ &= \frac{t^3}{3!},\ 0 \leq t \leq 1. \end{aligned}$$

Again we see a pattern developing, so conjecturing that $F_{T_n}(t) = \frac{t^n}{n!}$ for $0 \le t \le 1$, we establish that

$$\begin{aligned} F_{T_{n+1}}(t) &= \int_0^t F_{T_n}(t-s) f_X(s) ds \\ &= \int_0^t \frac{(t-s)^n}{n!} ds \\ &= \frac{t^{n+1}}{(n+1)!}, \; 0 \le t \le 1 \end{aligned}$$

which proves the conjecture. The expected number of renewals by time t for $0 \le t \le 1$ for this uniform renewal process is thus given by

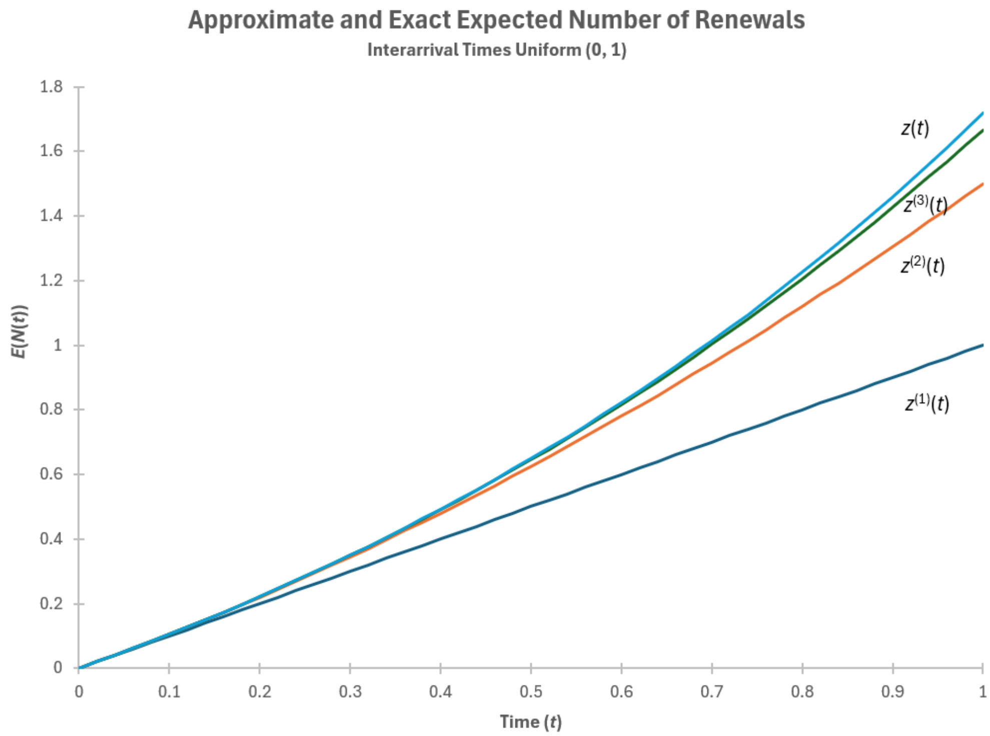
$$E(N(t)) = \sum_{n=1}^{\infty} \frac{t^n}{n!} = e^t - 1.$$

We can verify that this expression satisfies the renewal equation for $E(N(t))$ by evaluating

$$\begin{aligned} E(N(t)) &= F_{T_1}(t) + \int_0^t E(N(t-s)) f_X(s) ds \\ &= t + \int_0^t (e^{t-s} - 1) ds \\ &= t + e^t - t - 1 \\ &= e^t - 1 \end{aligned}$$

as was to be shown.

In this example, for $0 \le t \le 1$ it was possible to obtain the exact value for $E(N(t))$ analytically, but this will not always be possible. However, numerical approximations can always be obtained via our successive approximation scheme. The figure below plots $z^{(1)}(t)$, $z^{(2)}(t)$, $z^{(3)}(t)$, and the exact value $z(t)$ for the uniform example considered above. As shown in the figure, an excellent approximation to $z(t) = E(N(t))$ is provided by just the first three iterations of our successive approximation scheme.



3.2. Example: The Poisson process

Perhaps the best known renewal process is the Poisson process. In the Poisson process, the interarrival times are exponentially distributed with mean $1/\lambda$ where $\lambda > 0$ is the *arrival rate*, that is, the interarrival times have density

$$f_X(s) = f_{T_1}(s) = \lambda e^{-\lambda s},\ s \geq 0$$

and cumulative distribution

$$\Pr\{X \leq t\} = F_{T_1}(t) = \int_0^t \lambda e^{-\lambda s} ds = 1 - e^{-\lambda t}.$$

It is of course well-known that the probability distribution of the number of arrivals (or number of renewals) that occur by time t (starting from time 0) is given by the Poisson distribution

$$\Pr\{N(t) = n\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t},\ n = 0, 1, 2, \ldots;\ t \geq 0$$

and from this probability distribution the expected number of renewals follows immediately as

$$E(N(t)) = \sum_{n=0}^{\infty} n \times \frac{(\lambda t)^n}{n!} e^{-\lambda t} = \lambda t.$$

Since the Poisson process is also a renewal process, it must be true that the expected number of renewals also satisfies the renewal equation! Verification is simple:

$$\begin{aligned} E(N(t)) &= F_{T_1}(t) + \int_0^t E(N(t-s)) f_X(s) ds \\ &= 1 - e^{-\lambda t} + \int_0^t \lambda(t-s) \times \lambda e^{-\lambda s} ds \\ &= 1 - e^{-\lambda t} + \lambda t + e^{-\lambda t} - 1 \\ &= \lambda t \end{aligned}$$

as was to be shown.

4. The Renewal Density $h(t)$

The renewal density $h(t)$ is defined as the derivative of the expected number of renewals, that is,

$$h(t) \equiv \frac{d}{dt} E(N(t)).$$

The interpretation of the renewal density is that the probability a renewal occurs in a time interval $(t, t + \Delta t)$ is given by $h(t)\Delta t$. For the Poisson process, where $E(N(t)) = \lambda t$, the renewal density $h(t) = \lambda$, the rate of the Poisson process. For the uniform (0, 1) renewal process operating on $0 \leq t \leq 1$, we discovered earlier that $E(N(t)) = e^t - 1$, and thus the renewal density $h(t) = e^t$ for this process.

Working in reverse, it is clear that the renewal density integrates to the expected number of renewals, that is, $\int_0^t h(s)ds = E(N(t))$. To better understand this relationship, we argue informally as follows: define the binary variable $B(t) = 1$ if there is a renewal at time t, and 0 otherwise. Then the total number of renewals that occur between 0 and t is given by

$$N(t) = \int_0^t B(s) ds.$$

Now $B(t)$ is a Bernoulli random variable, which means that the expected value of $B(t)$ equals the probability of a renewal at time t. It follows that

$$\begin{aligned} E(N(t)) &= E(\int_0^t B(s)ds) \\ &= \int_0^t E(B(s))ds \\ &= \int_0^t h(s)ds \end{aligned}$$

since $\Pr\{B(t) = 1\} = h(t)\Delta t$.

What else can we say about the renewal density? Recalling that

$$E(N(t)) = F_{T_1}(t) + \int_0^t E(N(t-s))f_X(s)ds,$$

taking the derivative yields

$$\begin{aligned} h(t) &= \frac{d}{dt}E(N(t)) \\ &= \frac{d}{dt}(F_{T_1}(t) + \int_0^t E(N(t-s))f_X(s)ds) \\ &= f_{T_1}(t) + \int_0^t h(t-s)f_X(s)ds. \end{aligned}$$

This is another renewal equation, but now expressed in terms of the renewal density instead of the expected number of renewals. This is a special case of the general renewal equation obtained by setting $g(t) = f_{T_1}(t)$ and recognizing $z(t)$ as the renewal density $h(t)$. We can easily verify this equation for the Poisson process where $h(t) = \lambda$ and $f_X(t) = \lambda e^{-\lambda t}$ for

$$\begin{aligned} & f_{T_1}(t) + \int_0^t h(t-s)f_X(s)ds \\ = \;& \lambda e^{-\lambda t} + \int_0^t \lambda \times \lambda e^{-\lambda s}ds \\ = \;& \lambda e^{-\lambda t} + \lambda(1 - e^{-\lambda t}) \\ = \;& \lambda. \end{aligned}$$

Similarly, for the uniform (0, 1) renewal process operating on $0 \le t \le 1$ we have $h(t) = e^t$ and $f_X(t) = 1$; plugging into the renewal equation for the renewal density we obtain

$$\begin{aligned} & f_{T_1}(t) + \int_0^t h(t-s) f_X(s) ds \\ = & \ 1 + \int_0^t e^{t-s} ds \\ = & \ 1 + e^t - 1 \\ = & \ e^t. \end{aligned}$$

What else can we say about the renewal density? Recall that we previously determined that

$$E(N(t)) = \sum_{n=1}^{\infty} F_{T_n}(t).$$

Since the renewal density $h(t)$ is just the derivative of the expected number of renewals, we immediately obtain

$$\begin{aligned} h(t) &= \frac{d}{dt} E(N(t)) \\ &= \sum_{n=1}^{\infty} f_{T_n}(t). \end{aligned}$$

Now it should be crystal clear why $h(t)\Delta t$ represents the probability that a renewal occurs in the time slice $(t, t + \Delta t)$: since $f_{T_n}(t)\Delta t$ is the probability that the n^{th} renewal occurs in $(t, t + \Delta t)$, summing $f_{T_n}(t)\Delta t$ over all n yields the probability that *some* renewal occurs in $(t, t + \Delta t)$! This is just another way of saying that $h(t)\Delta t$ is the probability that a renewal occurs within $(t, t + \Delta t)$.

5. Solving the General Renewal Equation

Recall the definition of the general renewal equation: given a known deterministic function $g(t)$ and known interarrival time distribution $f_X(t)$, the general renewal equation yields that function $z(t)$ that satisfies

$$z(t) = g(t) + \int_0^t z(t-s) f_X(s) ds.$$

To solve this equation, we will utilize the same successive approximation scheme we used earlier to determine $E(N(t))$. To recap, we start by defining $z^{(0)}(t) = 0$, and then iterate according to the schedule

$$z^{(i+1)}(t) = g(t) + \int_0^t z^{(i)}(t-s) f_X(s) ds$$

and search for a pattern that we can then verify by substitution. First, we define functions

$$\phi_j(t) \equiv \int_0^t g(t-s) f_{T_j}(s) ds.$$

Now begin iterating. Clearly $z^{(1)}(t) = g(t)$, so now we evaluate

$$\begin{aligned} z^{(2)}(t) &= g(t) + \int_0^t z^{(1)}(t-s) f_X(s) ds \\ &= g(t) + \int_0^t g(t-s) f_X(s) ds \\ &= g(t) + \phi_1(t) \end{aligned}$$

for recall that the random variables X and T_1 have the same probability distributions. Pressing ahead to the next iterate, we have

$$\begin{aligned} z^{(3)}(t) &= g(t) + \int_0^t z^{(2)}(t-s) f_X(s) ds \\ &= g(t) + \int_0^t (g(t-s) + \phi_1(t-s)) f_X(s) ds. \end{aligned}$$

We immediately recognize $\int_0^t g(t-s) f_X(s) ds = \phi_1(t)$ so it remains to evaluate

$$\begin{aligned} & \int_0^t \phi_1(t-s) f_X(s) ds \\ = & \int_{s=0}^t \left\{ \int_{u=0}^{t-s} g(t-s-u) f_{T_1}(u) du \right\} f_X(s) ds. \end{aligned}$$

Make the substitution $w = s + u$, which implies that $u = w - s$. With this substitution,

$$\begin{aligned}
&\int_{s=0}^{t} \left\{ \int_{u=0}^{t-s} g(t-s-u) f_{T_1}(u) du \right\} f_X(s) ds \\
= &\int_{w=0}^{t} g(t-w) \left\{ \int_{s=0}^{w} f_{T_1}(w-s) f_X(s) ds \right\} dw \\
= &\int_{w=0}^{t} g(t-w) f_{T_2}(w) dw \\
= &\ \phi_2(t).
\end{aligned}$$

We conclude that

$$z^{(3)}(t) = g(t) + \phi_1(t) + \phi_2(t).$$

This is enough to suggest a pattern, namely

$$z^{(j+1)}(t) = g(t) + \sum_{i=1}^{j} \phi_i(t),$$

which leads to a conjecture for the overall solution to the general renewal equation, namely

$$z(t) = g(t) + \sum_{i=1}^{\infty} \phi_i(t).$$

Now, note that

$$\begin{aligned}
\sum_{i=1}^{\infty} \phi_i(t) &= \sum_{i=1}^{\infty} \int_0^t g(t-s) f_{T_i}(s) ds \\
&= \int_0^t g(t-s) \left\{ \sum_{i=1}^{\infty} f_{T_i}(s) \right\} ds \\
&= \int_0^t g(t-s) h(s) ds.
\end{aligned}$$

We have thus arrived at an amazing result: the solution to the general renewal equation is given by

$$z(t) = g(t) + \int_0^t g(t-s) h(s) ds.$$

The solution depends upon two functions: $g(t)$ (which is presumed known), and the renewal density $h(t)$ (which can be evaluated by different methods as discussed earlier). We leave it as an exercise for the reader to show that the solution above verifies the original general renewal equation.

Consider again the renewal equation for the expected number of renewals, $E(N(t))$. That equation sets $g(t) = F_{T_1}(t)$, so substituting this while writing the renewal density $h(t)$ as $\sum_{n=1}^{\infty} f_{T_n}(t)$ in the general renewal equation yields

$$
\begin{aligned}
E(N(t)) &= F_{T_1}(t) + \int_0^t F_{T_1}(t-s) \left\{ \sum_{n=1}^{\infty} f_{T_n}(s) \right\} ds \\
&= F_{T_1}(t) + \sum_{n=2}^{\infty} F_{T_n}(t) \\
&= \sum_{n=1}^{\infty} F_{T_n}(t)
\end{aligned}
$$

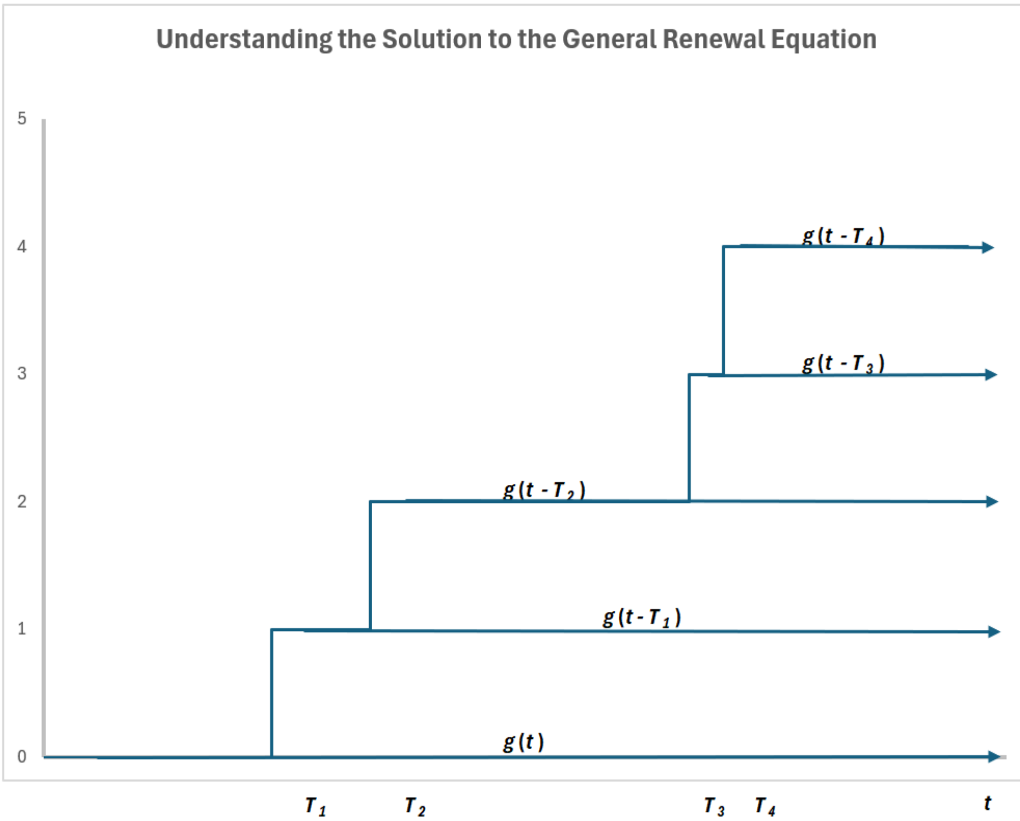
as shown previously.

5.1. Understanding the Solution to the General Renewal Equation

Imagine initiating a series of investments starting a times $0, T_1, T_2, T_3, T_4, ...$ From the time an investment is initiated, revenue accrues according to the function $g(s)$ where s is the *elapsed* time from the start of the investment. An initial investment is made at time 0, but afterwards investments are made in accord with a renewal process – each new renewal corresponds to the initiation of a new investment. Now ask, at some time $t > 0$, how much revenue has been accrued in total over all investments? Since the elapsed time from the initiation of an investment at time T_i until time t is just $t - T_i$, it must be that the total revenue accrued starting from the beginning investment at time 0 until time t is given by the random quantity

$$
Z(t) = g(t) + \sum_{i=1}^{\infty} g(t - T_i)^+
$$

where $g(s)^+ = g(s)$ for $s \geq 0$ and 0 otherwise. The figure below illustrates the situation when four renewals occur before time t, in which case the total revenue accrued is given by $Z(t) = g(t) + \sum_{i=1}^{4} g(t - T_i)$ (we don't need the "+" superscript as we know that all four renewals happened before time t.



The general renewal equation can thus be thought of as finding the *expected* total revenue generated from investments that start at times 0 and then at each renewal epoch afterwards up until time t where the expectation is over the times of the renewal epochs. That is,

$$
\begin{aligned}
E(Z(t)) \equiv z(t) &= g(t) + E[\sum_{i=1}^{\infty} g(t - T_i)^+] \\
&= g(t) + \sum_{i=1}^{\infty} \int_0^t g(t-s) f_{T_i}(s) ds \\
&= g(t) + \int_0^t g(t-s) h(s) ds.
\end{aligned}
$$

5.1.1. Example: $g(t) = rt$

Suppose that revenue accrues at a constant rate r from the time of investment. Then

$$z(t) = r \times t + \int_0^t r \times (t - s)h(s)ds.$$

If the renewal process in question is a Poisson process, then $h(s) = \lambda$, the arrival rate of the process, and we have

$$z(t) = rt + \lambda r \frac{t^2}{2}.$$

If on the other hand the renewal process has uniformly distributed interarrival times between 0 and 1, then for $0 \leq t \leq 1$ we have $h(s) = e^s$ and thus

$$\begin{aligned} z(t) &= rt + \int_0^t r \times (t - s)e^s ds \\ &= rt - r\left(t - e^t + 1\right) \\ &= r(e^t - 1). \end{aligned}$$

Since the mean interarrival time for the uniform renewal process just described is equal to 1/2, a Poisson process with the same mean interarrival time would have $\lambda = 2$, and for $0 \leq t \leq 1$ such a process would generate $rt + rt^2 \geq r(e^t - 1)$ revenue in total. A Poisson process with the same mean interarrival time as a uniform (0, 1) renewal process would generate more revenue in expectation.

5.1.2. Example: $g(t) = e^{rt}$

Imagine a terrorist organization that initializes bioterror attacks in different locations at times given by a renewal process. Further, imagine that the number of persons infected following the initiation of an attack grows exponentially with rate r, and thus the number of infected persons s time units after an attack equals e^{rs}. The expected total number of infected persons over all attacks at time t would then follow

$$z(t) = e^{rt} + \int_0^t e^{r(t-s)}h(s)ds.$$

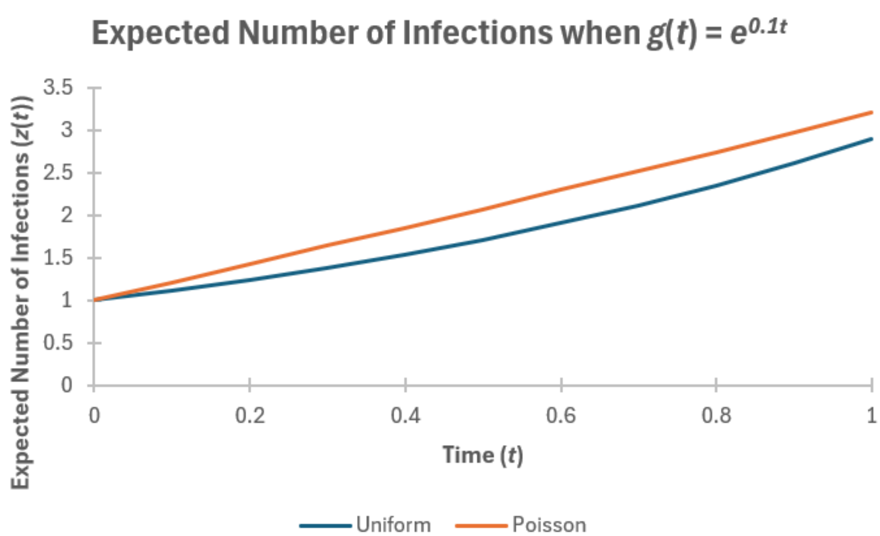
If the renewal process initiating attacks was Poisson with rate λ, the expected number of infected persons by time t would then be given by

$$\begin{aligned} z(t) &= e^{rt} + \int_0^t e^{r(t-s)} \lambda ds \\ &= e^{rt} + \frac{\lambda}{r}(e^{rt} - 1). \end{aligned}$$

If instead the renewal process was uniform (0, 1), then for $0 \le t \le 1$ the expected total number of infected person by time t would equal

$$\begin{aligned} z(t) &= e^{rt} + \int_0^t e^{r(t-s)} e^s ds \\ &= e^{rt}(1 + \int_0^t e^{(1-r)s} ds) \\ &= e^{rt}(1 + \frac{1}{1-r}(e^{(1-r)t} - 1)). \end{aligned}$$

The expected number of infections over time for the Poisson and uniform renewal processes when $r = 0.1$ and $\lambda = 2$ (so the uniform and Poisson processes have the same mean interarrival times) are shown in the graph below.



5.1.3. Finding $E(N(t)^2)$

To find the variance of the number of renewals in a given time period, one first computes the mean square $E(N(t)^2)$, and then subtracts the squared mean $E(N(t))^2$. Recall that if S is the time of the first renewal, then

$$N(t) = 1 + N(t - S).$$

We can use this relationship to write

$$\begin{aligned} E(N(t)^2) &= E[(1 + N(t-S))^2] \\ &= 1 + 2E(N(t-S)) + E(N(t-S)^2). \end{aligned}$$

To evaluate the expectation on the right hand side above, we must integrate over the density $f_X(s)$ which yields

$$\begin{aligned} E(N(t)^2) &= \int_0^t \left(1 + 2E(N(t-s)) + E(N(t-s)^2)\right) f_X(s)ds \\ &= \int_0^t \left(2 + 2E(N(t-s)) - 1 + E(N(t-s)^2)\right) f_X(s)ds \\ &= 2E(N(t)) - F_{T_1}(t) + \int_0^t E(N(t-s)^2) f_X(s)ds. \end{aligned}$$

This is a renewal equation with $z(t) = E(N(t)^2)$ and $g(t) = 2E(N(t)) - F_{T_1}(t)$. Can you deduce the solution for $z(t)$ via the general renewal equation? Can you verify your solution by taking advantage of the facts that $\Pr\{N(t) \geq n\} = \Pr\{T_n \leq t\}$, and

$$E(N(t)^2) = \sum_{n=0}^{\infty} n^2 \Pr\{N(t) = n\}?$$

I leave that as an exercise for the reader.

6. Asymptotic Results for $N(t)$ and $h(t)$

By now the reader should be convinced that there is a rich theory underlying renewal processes, with the general renewal equation providing a unifying approach to formulating the expected value of desired functions defined on renewal processes. However, other than in special cases where the probability densities and/or distributions of $f_{T_n}(t)/F_{T_n}(t)$ can be calculated explicitly (examples include the Poisson and uniform renewal processes), analytical results are not forthcoming for finite values of time. Numerical calculations, however, can always be obtained via the successive approximation scheme introduced for the general renewal equation, that is, by setting $z^{(0)}(t) = 0$ and iterating in accordance with

$$z^{(i+1)}(t) = g(t) + \int_0^t z^{(i)}(t-s) f_X(s) ds.$$

One iterates until successive approximations $z^{(i)}(t)$ and $z^{(i+1)}(t)$ are sufficiently close. In this way, one can solve for the moments of the number of renewals, or the expected value of different functions deriving from carefully chosen $g(t)$ functions.

However, the analytical difficulty of proceeding with all but a select few renewal processes should help sober the reader regarding why analytically tractable processes such as the Poisson are often the default choice in operations research models. This same difficulty might also cause the reader to ask whether there are simpler asymptotic results for renewal processes that work well as t gets large. We will now focus on this second set of questions, and show that there are very easy-to-use asymptotic results that apply to the distribution of $N(t)$, and consequently to the renewal density $h(t)$.

To begin, recall the random walk definition of the time of the n^{th} renewal epoch T_n

$$T_n = \sum_{i=1}^{n} X_i$$

where all of the X_i's are iid as random variable X, the interarrival time for the renewal process. As n gets large, the central limit theorem assures us that $T_n \longrightarrow N(n\tau, n\sigma^2)$, that is, the distribution of T_n tends towards a normal distribution with mean $n\tau$ and variance $n\sigma^2$ where τ and σ^2 are the mean and variance of the interarrival time X.

What about the number of renewals $N(t)$? It should come as no surprise that

$$\lim_{t \to \text{BIG}} E(N(t)) = \frac{t}{\tau}$$

for arguing informally, if the expected time between renewals equals τ, then the expected total time due to $N(t)$ renewals should (roughly) equal $\tau E(N(t)) = t$.

An immediate consequence of this is that

$$\begin{aligned}\lim_{t\to\text{BIG}} h(t) &= \lim_{t\to\text{BIG}} \frac{d}{dt}E(N(t)) \\ &= \lim_{t\to\text{BIG}} \frac{d}{dt}\left(\frac{t}{\tau}\right) \\ &= \frac{1}{\tau}.\end{aligned}$$

This is pretty consequential – as $t \to$ BIG, no matter the specifics of the underlying interarrival time distribution that drives the renewal process, the renewal density approaches the constant $1/\tau$, also known as the *rate* of the renewal process. Far enough in the future, all renewal processes have that Poisson feel in that the probability of a renewal in any interval $(t, t+\Delta t)$ just equals $\Delta t/\tau$. Note that for the Poisson process, $\tau = 1/\lambda$ and thus $h(t) = \lambda$ for *all* t.

But what of the probability distribution for $N(t)$ as t becomes large? Recall the crucial event equivalence of renewal processes:

The events $N(t) \geq n$ and $T_n \leq t$ are the same events!

which implies that

$$\Pr\{N(t) \geq n\} = \Pr\{T_n \leq t\}.$$

The central limit theorem implied normality for T_n allows us to write (with slight abuse of notation in letting Z represent the standard normal random variable)

$$\begin{aligned}\Pr\{T_n \leq t\} &\approx \Pr\{Z \leq \frac{t - E(T_n)}{\sqrt{Var(T_n)}}\} \\ &= \Pr\{Z \leq \frac{t - n\tau}{\sigma\sqrt{n}}\} \\ &\equiv \Phi(\frac{t - n\tau}{\sigma\sqrt{n}})\end{aligned}$$

where $\Phi(\cdot)$ denotes the standard normal cdf. In turn this suggests writing

$$\Pr\{N(t) \geq n\} \approx \Phi(\frac{t - n\tau}{\sigma\sqrt{n}})$$

which due to the symmetry of the standard normal distribution about zero is equivalent to writing

$$\begin{aligned}\Pr\{N(t) \le n\} &\approx \Phi(\frac{n\tau - t}{\sigma\sqrt{n}}) \\ &= \Phi(\frac{n - t/\tau}{(\sigma/\tau)\sqrt{n}}).\end{aligned}$$

The problem with this latter expression, however, is that it does not define a constant-parameter probability distribution for $N(t)$ – the appearance of $\sqrt{n}$ in the denominator ruins the idea that there is a stable variance for $N(t)$. The problem can be immediately recognized – the values of n that $N(t)$ can assume must scale with t (just like the values of t that T_n can assume essentially scale with n via $StDev(T_n) = \sigma\sqrt{n}$).

To fix this, for some arbitrary constant η we define

$$\begin{aligned}n_t &= \frac{t}{\tau} + \eta\sigma\sqrt{\frac{t}{\tau^3}} \\ &= \frac{t}{\tau}(1 + \eta\sigma\sqrt{\frac{1}{t\tau}}).\end{aligned}$$

This appears arbitrary, but watch what happens – with this value of n_t we consider the time of the n_t^{th} renewal, T_{n_t}, and write

$$\Pr\{T_{n_t} \le t\} = \Pr\{Z \le \frac{t - n_t\tau}{\sigma\sqrt{n_t}}\}.$$

Now

$$\begin{aligned}t - n_t\tau &= t - \tau\left[\frac{t}{\tau}(1 + \eta\sigma\sqrt{\frac{1}{t\tau}})\right] \\ &= t - t - \eta\sigma t\sqrt{\frac{1}{t\tau}} \\ &= -\eta\sigma\sqrt{\frac{t}{\tau}}.\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\frac{t - n_t\tau}{\sigma\sqrt{n_t}} &= -\frac{\eta\sigma\sqrt{\frac{t}{\tau}}}{\sigma\sqrt{n_t}} \\
&= -\frac{\eta\sqrt{\frac{t}{\tau}}}{\sqrt{\frac{t}{\tau}(1+\eta\sigma\sqrt{\frac{1}{t\tau}})}} \\
&= -\frac{\eta}{\sqrt{(1+\eta\sigma\sqrt{\frac{1}{t\tau}})}}
\end{aligned}$$

Taking the limit as $t \to \mathrm{BIG}$ we obtain

$$\begin{aligned}
\lim_{t\to\mathrm{BIG}} \frac{t - n_t\tau}{\sigma\sqrt{n_t}} &= \lim_{t\to\mathrm{BIG}} -\frac{\eta}{\sqrt{(1+\eta\sigma\sqrt{\frac{1}{t\tau}})}} \\
&= -\eta \quad (!!!)
\end{aligned}$$

Now we can comfortably claim that

$$\begin{aligned}
\lim_{t\to\mathrm{BIG}} \Pr\{T_{n_t} \le t\} &= \lim_{t\to\mathrm{BIG}} \Pr\{Z \le \frac{t - n_t\tau}{\sigma\sqrt{n_t}}\} \\
&= \lim_{t\to\mathrm{BIG}} \Pr\{Z \le -\eta\} \\
&= \Phi(-\eta).
\end{aligned}$$

Finally we can write that

$$\begin{aligned}
\lim_{t\to\mathrm{BIG}} \Pr\{N(t) \ge n_t\} &= \lim_{t\to\mathrm{BIG}} \Pr\{T_{n_t} \le t\} \\
&= \Phi(-\eta)
\end{aligned}$$

and therefore

$$\lim_{t\to\mathrm{BIG}} \Pr\{N(t) \le n_t\} = \Phi(\eta)$$

again owing to the symmetry of the standard normal distribution about zero.

Almost there! Recall that since

$$n_t \equiv \frac{t}{\tau} + \eta\sigma\sqrt{\frac{t}{\tau^3}}$$

we recognize η as

$$\eta = \frac{n_t - \frac{t}{\tau}}{\sigma\sqrt{\frac{t}{\tau^3}}}.$$

But if $\lim_{t\to\text{BIG}} \Pr\{N(t) \le n_t\} = \Phi(\eta)$, and we know that $\lim_{t\to\text{BIG}} E(N(t)) = t/\tau$, we have just learned that

$$\lim_{t\to\text{BIG}} Var(N(t)) = \frac{\sigma^2}{\tau^3}t.$$

We have our asymptotic result for the number of renewals by time t – as $t \to$ BIG, $N(t)$ becomes normally distributed with mean t/τ and variance $(\sigma^2/\tau^3)t$. How cool is that???

6.1. Example: Poisson Process

For the Poisson process, you already know that for any t, $E(N(t)) = Var(N(t)) = \lambda t$ while the number of renewals is Poisson distributed. What does our asymptotic result say? The Poisson process derives from exponential interarrival times with mean $\tau = 1/\lambda$ and variance $\sigma^2 = 1/\lambda^2$. The asymptotic result says that as $t \to$ BIG,

$$\begin{aligned} E(N(t)) \stackrel{t\to\text{BIG}}{\to} & \quad t/\tau \\ = & \quad t/(1/\lambda) \\ = & \quad \lambda t. \end{aligned}$$

How about the variance? Our asymptotic result says that

$$\begin{aligned} Var(N(t)) \stackrel{t\to\text{BIG}}{\to} & \quad \frac{\sigma^2}{\tau^3}t \\ = & \quad \frac{(1/\lambda^2)}{(1/\lambda^3)}t \\ = & \quad \lambda t. \end{aligned}$$

So for the Poisson process, our asymptotic result says that the number of renewals by time t is normally distributed with mean and variance both equal to λt. And as you already know, the normal distribution provides an excellent approximation to the Poisson distribution if the Poisson mean is at least equal to 10.

6.2. Example: Uniform (0, 1) Process

If the interarrival times are uniformly distributed between 0 and 1, then the mean interarrival time is given by $\tau = 1/2$, while the variance of the interarrival time is equal to

$$\int_0^1 (x - \frac{1}{2})^2 dx = \frac{1}{12}.$$

All of our examples with this uniform process to this point have focused only properties of $N(t)$ for $0 \leq t \leq 1$ to keep the examples simple, but now we are allowing $t \to$ BIG! We see that asymptotically, for the uniform (0,1) process that

$$E(N(t)) \stackrel{t \to \text{BIG}}{\to} \frac{t}{\tau} = 2t$$

and

$$\begin{aligned} Var(N(t)) \stackrel{t \to \text{BIG}}{\to} & \frac{\sigma^2}{\tau^3} t \\ = & \frac{1/12}{1/8} t \\ = & \frac{2}{3} t. \end{aligned}$$

Compared to the Poisson process with $\lambda = 2$ (so both renewal processes have the same *rate*), we see that the number of renewals $N(t)$ for the uniform (0, 1) process is much less variable than the corresponding Poisson process.

7. Random Incidence

Suppose that a renewal process with interarrival times X distributed according to the pdf $f_X(x)$ has been operating for some time. Suppose further than an outside "observer" arrives at a time t chosen independently of the renewal process (the arrival of the observer does *not* constitute a renewal as the observer is "outside" the process). Let random variable G denote the duration of the interarrival time (or *gap*) that is entered by the randomly arriving observer. What can we say about this random variable?

The key observation is that the likelihood of entering a gap of duration g depends upon two things: the relative frequency with which such gaps occur (as determined by the interarrival pdf $f_X(g)$), and the duration of the gap g itself.

This enables us to write immediately a proportionality condition that the density of G must obey, namely

$$f_G(g) \propto g \times f_X(g),\ g \geq 0.$$

Since G is a random variable, its density must integrate to 1, which means that

$$\begin{aligned} f_G(g) &= \frac{g \times f_X(g)}{\int_0^\infty x \times f_X(x)dx} \\ &= \frac{g \times f_X(g)}{E(X)},\ g \geq 0. \end{aligned}$$

We have just derived the probability density function of the gap duration for a randomly arriving observer. We refer to this entire phenomenon as "random incidence" because our outside observer randomly intrudes on (or is incident to) the process.

From this density it is easy to derive the moments of that gap duration entered via random incidence. We have

$$\begin{aligned} E(G^k) &= \int_0^\infty g^k f_G(g)dg \\ &= \int_0^\infty g^k \times \frac{g \times f_X(g)}{E(X)} dg \\ &= \frac{E(X^{k+1})}{E(X)}. \end{aligned}$$

In particular, setting $k = 1$ we learn that

$$\begin{aligned} E(G) &= \frac{E(X^2)}{E(X)} \\ &= \frac{Var(X) + E(X)^2}{E(X)} \\ &= E(X) + \frac{Var(X)}{E(X)} \\ &\geq E(X). \end{aligned}$$

The expected duration of a gap entered via random incidence is *always* at least as large as the expected interarrival time for the corresponding renewal process.

We can also quickly obtain

$$\begin{aligned} Var(G) &= E(G^2) - E(G)^2 \\ &= \frac{E(X^3)}{E(X)} - \left(\frac{E(X^2)}{E(X)}\right)^2. \end{aligned}$$

7.1. Example: Uniform (0, 1) Renewal Process

Suppose that the interarrival times X are uniformly distributed between 0 and 1, which means that $f_X(x) = 1$ for $0 \leq x \leq 1$ and zero elsewhere. Then

$$\begin{aligned} E(X) &= \int_0^1 x dx = \frac{1}{2}, \\ E(X^k) &= \int_0^1 x^k dx = \frac{1}{k+1} \end{aligned}$$

and thus

$$\begin{aligned} E(G) &= \frac{E(X^2)}{E(X)} \\ &= \frac{1/3}{1/2} \\ &= \frac{2}{3} \end{aligned}$$

and

$$\begin{aligned} E(G^2) &= \frac{E(X^3)}{E(X)} \\ &= \frac{1/4}{1/2} \\ &= \frac{1}{2} \end{aligned}$$

from which we deduce

$$\begin{aligned} Var(G) &= E(G^2) - E(G)^2 \\ &= \frac{1}{2} - (\frac{2}{3})^2 \\ &= \frac{1}{18} \\ &\leq \frac{1}{12} = Var(X). \end{aligned}$$

Also, the density of the gap duration entered at random is given by

$$\begin{aligned} f_G(g) &= \frac{g f_X(g)}{E(X)} \\ &= \frac{g}{1/2} \\ &= 2g,\ 0 \le g \le 1. \end{aligned}$$

This density shows clearly that random incidence biases our observer towards longer gaps compared to the uniform interarrival times. It also explains why $Var(G) < Var(X)$ since the values of G are more concentrated around $E(G)$ than the values of the uniformly distributed random variable X are around $E(X)$.

7.2. Example: Poisson Process

Suppose that the interarrival times X are exponentially distributed with mean $1/\lambda$, i.e. $f_X(x) = \lambda e^{-\lambda x}$, $x \ge 0$. Then the pdf of a gap entered by random incidence is given by

$$\begin{aligned} f_G(g) &= \frac{g \times f_X(g)}{E(X)} \\ &= \frac{\lambda g e^{-\lambda g}}{1/\lambda} \\ &= \lambda^2 g e^{-\lambda g},\ g \ge 0. \end{aligned}$$

This is a second order Erlang density (equivalently, a gamma density with shape parameter $\alpha = 2$), which is also the density function for the *sum* of two iid exponential random variables, each with mean $1/\lambda$. From this characterization we immediately deduce that

$$\begin{aligned} E(G) &= 2E(X) \\ &= \frac{2}{\lambda} \end{aligned}$$

while

$$\begin{aligned} Var(G) &= 2Var(X) \\ &= \frac{2}{\lambda^2}. \end{aligned}$$

The reader can verify these results using the general formulas for $E(G)$ and $Var(G)$ reported above.

8. Recurrence Times

Continuing with the random incidence model where an outside observer randomly enters some interarrival gap in a renewal process that has been operating for a long time, we ask the following question: how much time elapsed from the most recent arrival in the renewal process to the moment of random incidence? We refer to this duration as a *backwards recurrence time* and denote it by X^*. What can we say about this random variable?

First, we know that the pdf for the duration of a gap (G) entered by random incidence is given by

$$f_G(g) = \frac{g \times f_X(g)}{E(X)},\ g \geq 0.$$

Conditional on entering a gap of duration $G = g$, the specific location within this gap where random incidence occurs is equally likely to be anywhere within the gap (remember, the random arrival time of the outside observer is independent of the renewal process). This then means that the conditional pdf of X^* given that $G = g$ is uniformly distributed between 0 and g, that is

$$f_{X^*}(x|G = g) = \frac{1}{g},\ 0 \leq x \leq g$$

and zero elsewhere. Unconditioning over G we obtain the *joint* pdf of X^* and G as

$$\begin{aligned} f_{X^*,G}(x, g) &= f_{X^*}(x|G = g) \times f_G(g) \\ &= \frac{1}{g} \times \frac{g \times f_X(g)}{E(X)} \\ &= \frac{f_X(g)}{E(X)},\ 0 \leq x \leq g. \end{aligned}$$

Integrating out to obtain the marginal pdf of the backwards recurrence time X^* yields

$$\begin{aligned} f_{X^*}(x) &= \int_{g=x}^{\infty} f_{X^*,G}(x, g)dg \\ &= \int_{g=x}^{\infty} \frac{f_X(g)}{E(X)}dg \\ &= \frac{\Pr\{X \geq x\}}{E(X)},\ x \geq 0. \end{aligned}$$

We know that this is a valid pdf that integrates to 1 for

$$E(X) = \int_0^\infty \Pr\{X \geq x\} dx$$

for any (proper) non-negative random variable X, a formula often referred to as "integrating the tail."

To obtain the moments of the backwards recurrence time X^*, we integrate by parts to evaluate

$$\begin{aligned} E(X^{*^k}) &= \int_0^\infty \frac{x \times \Pr\{X \geq x\}}{E(X)} dx \\ &= \frac{E(X^{k+1})}{(k+1)E(X)}. \end{aligned}$$

As an aside, the integration by parts proceeds by letting $u = \frac{\Pr\{X \geq x\}}{E(X)}$, $dv = x^k dx$, and then recognizing that $du = -\frac{f_X(x)}{E(X)} dx$ and $v = \frac{x^{k+1}}{k+1}$. In particular, for $k = 1$, we see that the mean backwards recurrence time is given by

$$E(X^*) = \frac{E(X^2)}{2E(X)}.$$

We could have obtained this result directly by noting that due to the randomness of the arrival time of random incidence,

$$E(X^*|G = g) = \frac{g}{2}$$

and consequently

$$\begin{aligned} E(X^*) &= \int_0^\infty \frac{g}{2} \times f_G(g) dg \\ &= \int_0^\infty \frac{g}{2} \times \frac{g \times f_X(g)}{E(X)} dg \\ &= \frac{E(X^2)}{2E(X)} \end{aligned}$$

as claimed.

8.1. Example: Poisson Process

For the Poisson process, interarrival times are exponentially distributed with mean $1/\lambda$ and thus the pdf for the backwards recurrence time is given by

$$\begin{aligned} f_{X^*}(x) &= \frac{\Pr\{X \geq x\}}{E(X)} \\ &= \frac{e^{-\lambda x}}{1/\lambda} \\ &= \lambda e^{-\lambda x},\ x \geq 0. \end{aligned}$$

This is yet another manifestation of the "memoryless property" of the exponential distribution. The elapsed time from the previous renewal until the moment of random incidence has exactly the same exponential distribution as the interarrival time distribution itself!

8.2. Example: Uniform (0, 1) Process

If the interarrival times are uniformly distributed between 0 and 1, then the backwards recurrence time pdf is given by

$$\begin{aligned} f_{X^*}(x) &= \frac{\Pr\{X \geq x\}}{E(X)} \\ &= \frac{1-x}{1/2} \\ &= 2 \times (1-x),\ 0 \leq x \leq 1. \end{aligned}$$

8.3. Forwards Recurrence Time

Suppose that we redefine X^*as representing the remaining time from the moment of random incidence to a renewal process until the *next* renewal occurs. This is referred to as the *forwards recurrence time*. A vivid example is to imagine a bus stop where the interarrival times of successive buses constitute a renewal process, and our observer arrives at a random time to catch the bus. The forward recurrence time X^* reports the waiting time from arrival until the next bus arrives.

One can again condition on entering an interarrival gap $G = g$ via random incidence, which as before results in a time of entry uniformly distributed between 0 and g. But if the time of entry is uniformly distributed, so is the remaining time until the next renewal! This means that exactly the same arguments used to derive the probability distribution for the backwards recurrence time apply to the forwards recurrence time. We thus have that the *remaining* time from random incidence until the next renewal occurs also has pdf

$$f_{X^*}(x) = \frac{\Pr\{X \geq x\}}{E(X)},\ x \geq 0.$$

In the bus stop interpretation, the expected waiting time until the next bus arrives is given by $E(X^*) = E(X^2)/2E(X)$.

If buses arrive according to a Poisson process, then the mean waiting time until the next bus is exactly the same as if one had literally just missed the bus. More generally, for the Poisson process, the forward recurrence time is distributed exactly as the underlying exponential interarrival times. Talk about a memoryless process!

8.4. Example: Expected Waiting Time in $M/G/1$ Queue

Here we show how understanding renewal theory provides an almost instant derivation of one of the most celebrated results in queueing theory. First, recall the $M/M/1$ queueing model where customers arrive according to a Poisson process with rate λ, service times are exponentially distributed with mean $1/\mu$, and the utilization $\rho \equiv \lambda/\mu < 1$ (the utilization is the probability that the server is busy). The expected waiting time W_q for a newly arriving customer to the $M/M/1$ queue is easily derived as

$$\begin{aligned} W_q &= \Pr\{\text{Server is busy}\} \times E(\text{Remaining Service Time for Customer in Service}) \\ &\quad + \frac{1}{\mu} \times E(\text{Number of Customers in Queue}) \end{aligned}$$

The expected number of customers in queue, L_q, is related to the expected waiting time in queue via Little's Theorem

$$L_q = \lambda W_q.$$

And, since the service times are exponential, the expected remaining service time for a customer in service *at the time of a randomly arriving new customer* is just the forward recurrence time on the service time distribution, which in the case of the exponential distribution equals $1/\mu$. Substituting these insights into our equation for W_q we have

$$W_q = \rho \times \frac{1}{\mu} + \frac{\lambda W_q}{\mu}$$

which solves to

$$W_q = \frac{\rho/\mu}{1-\rho}$$

as is well known.

Now let's dispense with the exponential service time assumption, and allow service times to be arbitrarily distributed in accord with some random variable S. Exactly the same logic as was used to derive W_q for the $M/M/1$ queue applies, except that now the expected remaining time for a customer in service corresponds to a forward recurrence time on the service process – if someone is already in service at the time a new customer arrives (which is at random with respect to the service process), then the remaining service time from when our new customer arrives is the same as a forward recurrence time on the service process! The mean forward recurrence time is just $E(S^*) = E(S^2)/2E(S)$ (and we take $E(S) = 1/\mu$ as before), and thus for the $M/G/1$ queue we have

$$\begin{aligned} W_q &= \Pr\{\text{Server is busy}\} \times E(\text{Remaining Service Time for Customer in Service}) \\ &\quad + E(S) \times E(\text{Number of Customers in Queue}) \\ &= \rho \times \frac{E(S^2)}{2E(S)} + E(S) \times \lambda W_q \end{aligned}$$

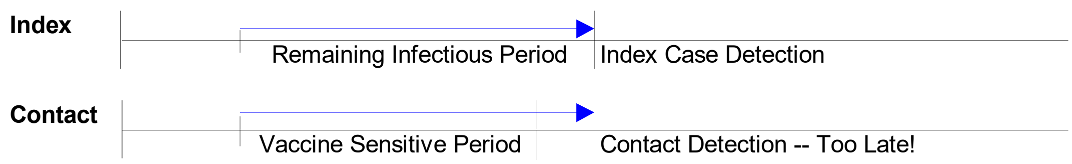
which solves to yield (upon noting that $\lambda E(S) = \lambda/\mu = \rho$)

$$\begin{aligned} W_q &= \frac{\rho}{1-\rho} \times \frac{E(S^2)}{2E(S)} \\ &= \frac{\lambda E(S^2)}{2(1-\rho)}. \end{aligned}$$

This is known as the *Pollaczek-Khinchine* formula, and is one of the most famous results in queueing theory.

8.5. The Race to Trace

Imagine an infectious disease that provides vaccination to exposed persons via contact tracing. You are likely familiar with this setup from the Covid-19 pandemic, but the operational aspects were worked out much earlier in the context of smallpox vaccination in response to a bioterrorism attack. Consider the diagram below:



The *index case* is infected at the leftmost time shown in the diagram; for convenience let this be at time $t = 0$. The index is interacting at random with other persons in the population. Suppose that the index infects a *contact* at a random time during the infectious period. If the duration of the infectious period is given by random variable X, then relative to the time the the contact is infected, the *remaining* time left in the index case's infectious period is the forward recurrence time X^*. Now, suppose that when a person is infected, they remain *vaccine sensitive* for duration V. Vaccine sensitivity means that if a newly infected person (in this case, the contact) is vaccinated within the vaccine sensitive period, they are protected from infection (or at least from serious consequences of disease, which for something like smallpox can be fatal). Now, the index case develops symptoms at the end of their infectious period, at which point contact tracing ensues. Optimistically presuming that contact tracing occurs at the speed of light (!), a contact infected by an index case will be vaccinated in time if $V \geq X^*$. This is to say, if the duration of the vaccine sensitive period for the contact exceeds the remaining time from infection until symptoms for the index, then the contact can be vaccinated in time. On the other hand, if $V < X^*$ (which is the situation shown in the diagram above), then even instantaneous vaccination of the contact upon the discovery that the index is infected (via the appearance of symptoms) will occur too late to save the contact.

We refer to this scenario as the *race to trace*, and seek the probability that a contact can be "saved" from infection with an index via (instantaneous) contact tracing and vaccination. Noting that V and X^* are independent, the probability of saving the contact is given by

$$
\begin{aligned}
\Pr\{\text{Save Contact}\} &= \Pr\{V \geq X^*\} \\
&= E_{X^*}(\Pr\{V \geq x | X^* = x\}) \\
&= \int_0^\infty \Pr\{V \geq x\} f_{X^*}(x) dx \\
&= \int_0^\infty \Pr\{V \geq x\} \times \frac{\Pr\{X \geq x\}}{E(X)} dx \\
&= \frac{1}{E(X)} \int_0^\infty \Pr\{V \geq x, X \geq x\} dx \\
&= \frac{1}{E(X)} \int_0^\infty \Pr\{\min(V, X) \geq x\} dx \\
&= \frac{E[\min(V, X)]}{E(X)}.
\end{aligned}
$$

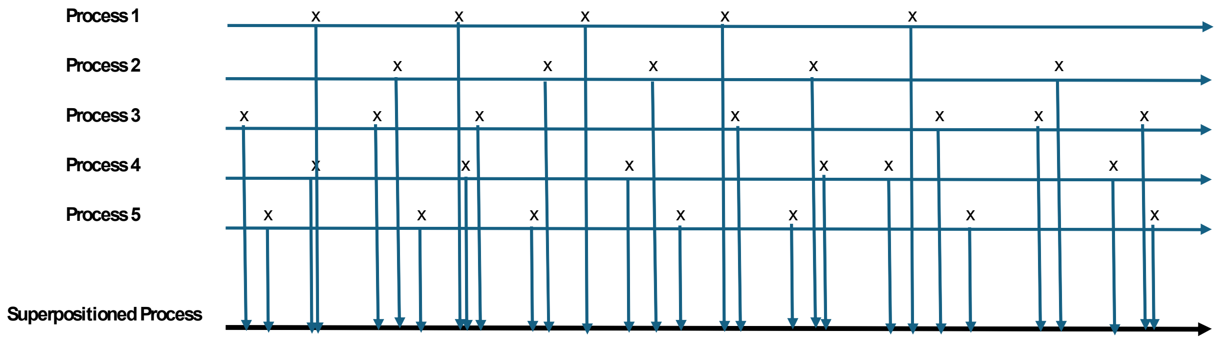
This is a very telling result. If $V << X$, then $\min(V, X) \approx V$ and Pr{Save Contact} $\approx E(V)/E(X)$ which will be a very small probability. At the other extreme, if $V >> X$, then $\min(V, X) \approx X$ and Pr{Save Contact} ≈ 1.

9. The Superposition of Renewal Processes

Suppose we have a basic renewal process with interarrival times X governed by pdf $f_X(x)$ and survivor function

$$
\begin{aligned}
S_X(x) &\equiv \Pr\{X \geq x\} \\
&= 1 - F_X(x),\ x \geq 0.
\end{aligned}
$$

Now imagine having n different renewal processes, each of which has interarrival times distributed as X. The superposition of these renewal processes is found by keeping track of the successive times at which each new arrival occurs without regard to which renewal process the arrival corresponds to. This situation is shown in the diagram below.



The diagram shows the arrival epochs for five different renewal processes (each arrival marked with an "x"), and the arrival epochs of the superpositioned process (the arrowheads along the bottom axis).

We are interested in the random variable Y, which denotes the durations of the interarrival times in the superpositioned process. It is immediately clear that the mean duration of the interarrival times in the superpositioned process is given by

$$E(Y) = \frac{E(X)}{n}$$

for if we are combining n independent processes, each with the same mean interarrival time $E(X)$, then the mean interarrival time duration in the superpositioned process must be $1/n^{\text{th}}$ as long. What else can we say about the distribution of Y? Let us define

$$G(y) = \Pr\{Y \geq y\}$$

as the survivor function for random variable Y. Suppose an outside observer was to arrive at the superpositioned process independently of the process itself. From the moment of random incidence to the superpositioned process, what is the time until the next superpositioned arrival, that is, what is the forward recurrence time Y^*? Clearly the pdf for this forward recurrence time must be given by

$$\begin{aligned} f_{Y^*}(y) &= \frac{\Pr\{Y \geq y\}}{E(Y)} \\ &= \frac{G(y)}{E(Y)} \\ &= \frac{G(y)}{E(X)/n} \\ &= \frac{n}{E(X)} \times G(y),\ y \geq 0. \end{aligned}$$

But, the time until the next superpositioned arrival from the moment of random incidence must also equal the minimum of the forward recurrence times for each of the constituent renewal processes – random incidence to the superpositioned process implies random incidence to *each* of the renewal processes being superpositioned! This means that

$$Y^* = \min_{1 \le j \le n} X_j^*$$

and as a consequence we must have

$$\begin{aligned}
\Pr\{Y^* \ge y\} &= \Pr\{\min_{1 \le j \le n} X_j^* \ge y\} \\
&= (\Pr\{X^* > y\})^n \\
&= \left(1 - \int_0^y f_{X^*}(x)dx\right)^n \\
&= (1 - \int_0^y \frac{S_X(x)}{E(X)}dx)^n.
\end{aligned}$$

An alternative expression for the pdf of Y^* can be obtained via the negative derivative of $\Pr\{Y^* \ge y\}$, that is

$$\begin{aligned}
f_{Y^*}(y) &= -\frac{d}{dy}\Pr\{Y^* \ge y\} \\
&= -\frac{d}{dy}(1 - \int_0^y \frac{S_X(x)}{E(X)}dx)^n \\
&= n(1 - \int_0^y \frac{S_X(x)}{E(X)}dx)^{n-1} \times \frac{S_X(y)}{E(X)} \\
&= \frac{n}{E(X)} \times (1 - \int_0^y \frac{S_X(x)}{E(X)}dx)^{n-1} \times S_X(y),\ y \ge 0.
\end{aligned}$$

Equating our two expressions for $f_{Y^*}(y)$ yields

$$\frac{n}{E(X)} \times G(y) = \frac{n}{E(X)} \times (1 - \int_0^y \frac{S_X(x)}{E(X)}dx)^{n-1} \times S_X(y)$$

which leaves us with

$$G(y) = S_X(y) \times (1 - \int_0^y \frac{S_X(x)}{E(X)}dx)^{n-1},\ y \ge 0$$

as the survival function for random variable Y.

It is convenient at this point to scale random variable Y by its mean, so define a new variable W by

$$\begin{aligned} W &= \frac{Y}{E(Y)} \\ &= \frac{n}{E(X)} \times Y. \end{aligned}$$

Then we have

$$\begin{aligned} \Pr\{W \geq w\} &= \Pr\{Y \geq \frac{E(X)}{n} w\} \\ &= G(\frac{E(X)}{n} w) \\ &= S_X(\frac{E(X)}{n} w) \times (1 - \int_0^{\frac{E(X)}{n} w} \frac{S_X(x)}{E(X)} dx)^{n-1}, \ w \geq 0. \end{aligned}$$

Let us take the limit as the number of constituent renewal processes $n \to \text{BIG}$. Note that

$$S_X(\frac{E(X)}{n} w) \stackrel{n \to \text{BIG}}{\to} S_X(0) = 1$$

while

$$\int_0^{\frac{E(X)}{n} w} \frac{S_X(x)}{E(X)} dx \stackrel{n \to \text{BIG}}{\to} \frac{S_X(0)}{E(X)} \times \frac{E(X)}{n} w = \frac{w}{n}.$$

Taken together we have that

$$\Pr\{W \geq w\} \stackrel{n \to \text{BIG}}{\to} (1 - \frac{w}{n})^{n-1} = e^{-w} \text{ (!!)}$$

Returning to our original random variable Y, the interarrival time in the superpositioned process, we have

$$\begin{aligned} \Pr\{Y \geq y\} = & \quad G(y) \stackrel{n \to \text{BIG}}{\to} \Pr\{W \geq \frac{n}{E(X)} y\} \\ = & \quad e^{-\frac{n}{E(X)} y}, \ y \geq 0. \end{aligned}$$

We have our result! As the number of renewal processes being superpositioned, n, gets large, the distribution of the time between arrivals in the superpositioned process tends towards an exponential distribution with mean $E(Y) = E(X)/n$ and variance $Var(Y) = (E(X)/n)^2$. If the original renewal processes had been Poisson processes with the same arrival rate λ, then the superpositioned process would also be a Poisson process with arrival rate $1/E(Y) = n/E(X) = n/(1/\lambda) = \lambda n$.

You can build further upon this result. Suppose you had c classes of renewal processes, and in each class j you superpositioned n_j iid renewal processes. Then you would reach an exponential limit for each of the c renewal classes. The superposition of all of the renewal processes across all classes would then also reach an exponential limit for the interarrival times in the superpositioned process, since the minimum of exponential random variables is also exponential, as you well know.

10. Additional Topics in Renewal Theory

We have exhausted the time we have allotted to the study of renewal processes, but there are a few additional topics you can explore on your own. If a renewal process begins with a special interarrival time that is different from all subsequent interarrival times, you obtain a *modified* renewal process. Suppose you are studying an equipment failure time problem. If the times between successive failures are iid from some distribution, starting from the time of a new machine, counting machine failures would constitute an *ordinary* renewal process of the type we have studied. But suppose you begin the process with a machine that has been in use for exactly a units of time. The remaining time until that machine fails depends upon the age a, and its distribution is of course determined by conditioning on the original failure time distribution. So you would have a first time to renewal given by this conditional remaining life given the machine has lived for exactly a, followed by regular renewal intervals. And as a very special case of a modified renewal process, suppose that you first encounter a machine that has been working for a *random* amount of time. The remaining life in this machine would thus be distributed as a forward recurrence time, after which all subsequent renewals would follow the regular process. A modified process that begins this way is called an *equilibrium* renewal process.

One can also study *alternating renewal processes*. Imagine a machine that is up or down (or imagine a patient who is in treatment or under observation). In the machine example, all up times are drawn from one probability distribution and on their own would constitute an ordinary renewal process. Similarly the down times are drawn from a different probability distribution and would on their own constitute a different ordinary renewal process. In an alternating renewal process, you switch between these two processes whenever there is a renewal –

the machine's up time is followed by down time, which is then followed by up time, which is then followed by down time, etc. Suppose that T_{up} and T_{down} are the expected times the machine stays up and down respectively. You arrive at a random time. What is the probability that the machine is up? Congratulate yourself if you guessed $E(T_{up})/E((T_{up}) + E(T_{down}))$.

There are many more extensions to the theory, but I hope these notes have given you some flavor of the beauty and applicability of renewal processes in operations research.

11. Some References

Everything contained in these notes is well-known to people well-versed in renewal theory, though not all of the topics have been explained as in these notes. However, if you are interested in learning more, or otherwise seeing standard presentations of renewal theory, please consult the following references (there are many more, the first reference by Cox is the classic).

David R. Cox (1962), *Renewal Theory*, Methuen: London.

Sheldon M. Ross (2014), *Introduction to Probability Models* (11th edition), Elsevier Academic Press: Oxford.

U. Narayan Bhat and Gregory K. Miller (2002), *Elements of Applied Stochastic Processes* (3rd Edition), John Wiley and Sons: New York.

Willy Feller (1941), On the integral equation of renewal theory. *Annals of Mathematical Statistics* **12**(3); 243-267.

William Feller (1966). *An Introduction to Probability Theory and its Applications: Volume II* (see Chapter 11 – Renewal Theory). John Wiley and Sons: New York.