EMD 630/MGMT 711
Summary of Results: Stable Population Model

Definitions

N = stable population size

$N(a)$ = population density at age a

$N(0)$ = aggregate arrivals (e.g. total births) per unit time

X, Y, Z = number susceptible, infected and infectious, and recovered respectively

$X(a), Y(a), Z(a)$ = age density of susceptibles, infecteds, and recovered respectively

L = duration of life

$\mu(a)$ = age-specific mortality rate (deaths per person per unit time at age a)

Note: $N(0) = \int_0^\infty N(a)\mu(a)da$, that is, total arrivals = total departures (both per unit time)

$E(L) = \ell$ = life expectancy

T_S = duration of time spent susceptible absent competing removal risk of death (i.e. T_S is how much time one would spend susceptible if one lived forever!)

λ = incidence rate of infection (infections per susceptible per unit time, e.g. infections per 100 person years)

T_I = duration of infection (and infectiousness) absent competing removal risk of death (i.e. T_I is how much time one would spend infected and infectious if one lived forever!)

υ = recovery rate from infection (recoveries per infected per unit time)

Note: the incidence rate could depend on age, while the recovery rate could depend on age, time spent infectious, or both; see the "system of flow" notes for the gory details

R_0 = reproductive number reporting the mean number of infections generated by a single infected person in a population of susceptibles

β = transmission rate per person per unit time (used in strong homogeneous (i.e. free) mixing models)

	General Case	Useful Approximation $T_S + T_I << L$	Anderson and May Useful Approx + Constant Incidence + Constant Recovery Rate + Type I Mortality
N	$N(0)E(L)$	$N(0)E(L)$	$N(0)\ell$
X	$N(0)E[\min(T_S, L)]$	$N(0)E(T_S)$	$N(0)/\lambda$
Y	$N(0)\Pr\{T_S < L\}E[\min(T_I, L - T_S \| T_S < L)]$	$N(0)E(T_I)$	$N(0)/\upsilon$
Z	$N - X - Y$	$N - X - Y$	$N - X - Y$
$N(a)$	$N(0)\Pr\{L > a\}$	$N(0)\Pr\{L > a\}$	$N(0)$ for $0 \leq a < \ell; 0$ otherwise
$X(a)$	$N(0)\Pr\{L > a\}\Pr\{T_S > a\}$	$N(0)\Pr\{T_S > a\}$	$N(0)e^{-\lambda a}$
$Y(a)$	$N(0)\Pr\{L > a\}\Pr\{T_S \leq a < T_S + T_I\}$	$N(0)\Pr\{T_S \leq a < T_S + T_I\}$	$N(0)\frac{\lambda}{\lambda-\upsilon}(e^{-\upsilon a} - e^{-\lambda a})$
$Z(a)$	$N(a) - X(a) - Y(a)$	$N(a) - X(a) - Y(a)$	$N(a) - X(a) - Y(a)$
Mean Age at Infection	$E(T_S \| T_S < L)$	$E(T_S)$	$1/\lambda$
$R_0 = N/X$	$E(L)/E[\min(T_S, L)]$	$E(L)/E(T_S)$	$\lambda\ell$
R_0 (free mixing)	$N\beta E[\min(T_I, L^*)]$ (L^* is residual life)[1]	$N\beta E(T_I)$	$N\beta/\upsilon$

[1] The residual life is the remaining lifetime for someone selected at random in the disease free stable population – the idea is that when an infection is introduced, it is done by infecting a randomly selected susceptible from those present. L^* and L are not the same, as L is the duration of life from age 0 (birth), whereas L^* is the remaining life for a randomly selected person in a completely susceptible stable population.