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Sports Betting and Kelly Criterion

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Why Study Betting Models?

- Great opportunity to see probability in action, sometimes counterintuitive (e.g. pools)
- Pulls together models of team performance with market assessment of the same
- Aside from making (or more often, losing) money, provides an opportunity to see how market aggregates information to make predictions
- Applicable in many other domains (e.g. prediction markets)

Sports Betting 101

- Focus on binary events such as
  - A defeats B (moneyline bet)
  - A defeats B by more than *s* points (spread bet)
  - Together, A and B score more than *t* points (over/under or point totals bet)
- Can construct more complicated bets (parlays, teasers, etc.) from same principles

Straight Win-Loss Bet

- Suppose want to just bet on which team will win a game
- Commonly referred to as a money line bet
- In money line bet, the favored team is assigned a negative number (e.g. – 200) that indicates the stake required to win $100 (so you need to bet $200 to win $100, meaning that if favorite wins, you make $100 while if favorite loses, you lose $200)

Money Line Bets

- Similarly, the underdog gets a positive number (e.g. +150) that indicates what you win for a $100 bet if the favorite loses (meaning you lose $100 if the favorite wins, but gain $150 if there is an upset)

Let’s Look At Football Bets

- http://www.footballlocks.com/nfl_odds.shtml
- http://www.vegasinsider.com/nfl/odds/las-vegas/money/

Mixed Martial Arts?

- Consider Velasquez vs. Dos Santos for the UFC heavyweight title on October 19th
- Fought twice before, both times for the title, each winning once
- Velasquez is a wrestler and Dos Santos is a boxer
  - first fight, Dos Santos knocked out Velasquez in the first round to take the title
  - second fight, Velasquez dominated Dos Santos for five rounds to take a lop-sided decision
- http://www.bestfightodds.com/

Money Line Bets

- When does it make sense to take a position on a money line bet?
- Suppose you bet a favorite with a money line of $-f$ (f for favorite), and suppose you believe that favorite wins with probability a
- You expect a profit if

  $a \text{ x } 100 - (1 - a) \text{ x } f > 0$, or if

  $a > f / (100 + f)$
- In example, $f = 200$ so expect a profit if

  $a > 200 / (100 + 200) = 2/3$

Money Line Bets

- Suppose you bet the underdog, win u if underdog wins
- Again let a = Pr{Favorite wins} (according to you)
- Expect a profit if

  $(1 - a) \times u - a \times 100 > 0$, or if

  $a < u / (100 + u)$
- In example, $u = 150$ so expect profit if

  $a < 150 / (150 + 100) = 3/5$

Money Line Bets

- For same game and same sports book, you will always find $f > u$, as in our example:

  $f = 200 > 150 = u$

- Suppose $f < u$
- This would imply arbitrage!
  - Should bet favorite if $a > f / (100 + f)$
  - Should bet underdog if $a < u / (100 + u)$
  - If $f < u$, there will be a range of probabilities that support *both* wagers!

Money Line Arbitrage

- Will never happen inside a given sports book
- Could happen across sports books
- Suppose one book gives $f = 200$
- Another gives $u = 220$ for the same game!
- Place x bets on favorite in first book; y bets on underdog in second
- If favorite wins, you make \$100x – \$100y
- If upset, you make \$$uy$ – \$$fx$

Money Line Arbitrage

- If favorite wins, you make $\$100x - \$100y$
- If upset, you make $\$uy - \$fx$
  - Set them equal! Since $f = 200$; $u = 220$:

  $\$100x - \$100y = \$uy - \$fx$, or

  $\$(100 + f)\, x = \$(100 + u)\, y$, or

  $x / y = (100 + u) / (100 + f) = 320 / 300$, or

  $x = (320 / 300)\, y$
- Note: $\$100x - \$100y = \$100\,(320/300 - 1)\, y > 0$
- Profit = $\$(20/3)\, y$

How About Point Spread?

- Sports book posts spread, e.g. A –7, which means that team A is favored to win by *more* than 7 points (that is, "covering the spread" occurs if favorite wins by more than 7 points)
- Typically, point spread bet requires investment of \$11 for chance to win \$10
- Now let a be probability of covering spread
- Expect to profit if $10a - 11(1 - a) > 0$, or if

$$a > 11 / 21 = 0.5238$$

Same Rules Apply In Other Direction

- If think posted spread is too high, can pick underdog, but still have to pay \$11 to make \$10, so only sensible if you believe chance of favorite beating spread $< 10/21 = 0.4762$
- Sports book tries to set spread to get about half of the bets on each side
- No matter the outcome, sports book guaranteed profit (losers pay winners plus extra buck to the house!)

Over/Under or Totals Bet

- Now instead of betting on spread, one bets on total number of points scored
- Can take over or under posted amount
- Again usually bet $11 to win $10

Betting Pools

- Sometimes can enter competitions based on multiple games
  - e.g. football pool – each week, bet on winners for each NFL game; those who pick most winners win pool (share prize)
  - e.g. March Madness – fill out NCAA tournament bracket; whoever picks most correct games (or scores most points, where points awarded can follow any number of arcane/zany rules) wins (share prize)

Betting Pools More Complicated Than Single Bets!

- Imagine a single game where I think the “favorite” will win with probability *a* (for *actual*); $a > 1/2$
- I am betting against a single opponent who chooses that same team with probability *p* (for *pool* as we’ll soon consider more than one opponent); note that *p* is the probability that opponent chooses the team *I* think is the favorite, that is, *p* describes my opponent’s behavior
- We each put in $1; we each pick a team
- We are both right; both wrong; or one of us is right and the other is wrong

Which Team Should I Choose?

- Suppose I choose the favorite
  - If my opponent chooses the favorite (which happens wp p), I get my dollar back no matter what happens (either we are both right or both wrong)
  - If my opponent chooses the "dog" then I get both dollars if the favorite wins, and nothing if the favorite loses
  - So, if I choose the favorite, my expected take is

$$p \times 1 + (1 - p) \times a \times 2 = 2a + p - 2ap$$

Which Team Should I Choose?

- Suppose I choose the underdog
  - If my opponent also chooses the dog (wp $1 - p$) I get my dollar back no matter what (we are either both right or both wrong)
  - If my opponent chooses the favorite, I get both bucks if the favorite loses, and nothing if favorite wins
  - So, if I choose the dog, my expected take is

$$(1 - p) \times 1 + p \times (1 - a) \times 2 = 1 + p - 2ap$$

Which Team Should I Choose?

- So, I should choose the favorite if expected take given I choose favorite exceeds expected take given I choose the dog, or when

$$2a + p - 2ap > 1 + p - 2ap$$

- The above is true if $a > ½$ so no matter how my opponent plays, in this bet I should take the favorite!

Now Imagine Huge Number of Opponents

- Single game, n opponents (and n is huge)
- I have a favorite who I think wins with prob $a > ½$
- Each opponent will pick *my* favorite with probability p, and do so independently
- n is so large that there will surely be opponents who pick favorite, and opponents who pick dog
- Only way I can win anything is to pick correctly!

Huge Number of Opponents

- Suppose I pick the favorite
  - I get the right pick with probability a
  - Of course, expect np opponents to also get right pick
  - So my take is roughly a x $\frac{n+1}{1+np} \approx \frac{a}{p}$ for large n
- Suppose I pick the dog
  - I get right pick with probability $1 - a$
  - Expect $n(1 - p)$ opponents to also get right pick
  - So my take is roughly $(1 - a)$ x $\frac{n+1}{1+n(1-p)} \approx \frac{1-a}{1-p}$

Huge Number of Opponents

- So I should pick my favorite if

$$\frac{a}{p} > \frac{1-a}{1-p}$$

which is the same as

$$a > p$$

- Otherwise pick the dog
- This is called "betting the edge"
- Look familiar?
  - Money line: choose favorite if $a > f / (100 + f)$
  - Money line: choose dog if $a < u / (100 + u)$

Should Ever Pick Underdog When Everyone Else Does Too?

- Suppose two games; in each game your favorite wins with probability $a = ½ + \varepsilon$
- Huge pool, probability each opponent picks your favorite, $p = 0 + \varepsilon$
- How should I bet?
  - My favorite to win both games?
  - My favorite to win one game, dog the other?
  - Both dogs?

Weird Pool Continued

- Presume essentially all opponents take dog in each game
- If I take favorite in each game, chance I win both is basically ½ x ½ = ¼; essentially win $n/4$
- If win one, lose one, I'm no better than opponents
- If I take favorite in one game, dog in other, the chance I win game with favorite is ½ (and I'm sole winner), while no matter what happens in dog game I tie everyone else (who also pick dog), so overall winnings are essentially $n/2$
- Best to pick dog in one game, even though I know everyone else will!

Do Betting Markets Obviate Models That Predict Winners?

- If one has a really good model, might be able to get better info than sports books have (e.g. *a* vs *p* in pool examples)
- Flip view – betting market odds set by actions of real bettors with real money
- Maybe final posted odds are best measure of actual win probabilities
- Betting odds can capture events as they unfold, causing the line to move (markets after all)
- http://www.vegasinsider.com/nfl/odds/las-vegas/line-movement/patriots-@-falcons.cfm/date/9-29-13/time/2030#BT

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The Kelly Criterion

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When Gamblers Evaluate an Opportunity…

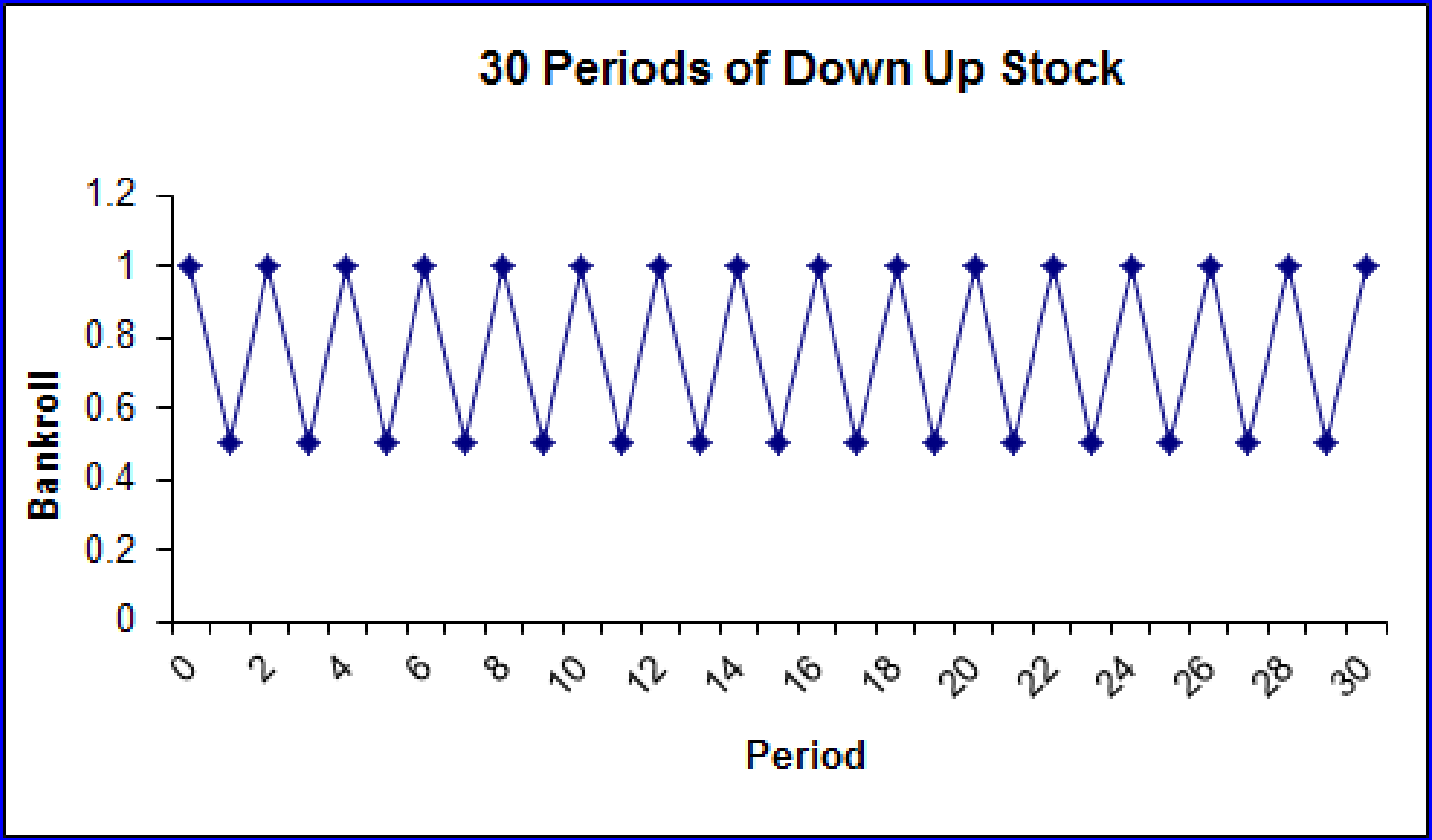
- Is the bet favorable (positive expected return)?
  - Key question – how does gambler's perceived probability of winning bet compare to chance implied by payoffs/losses
- If a bet is favorable, how much money should one wager?
- Same issues faced by financial investors:
  - Identify favorable investments
  - Determine amounts to invest

An Example

- Consider down up stock
  - Starting from current value, in the next period it is cut in half
  - But in period after that, it doubles in value to return to original level
- What happens if you just hold this stock?

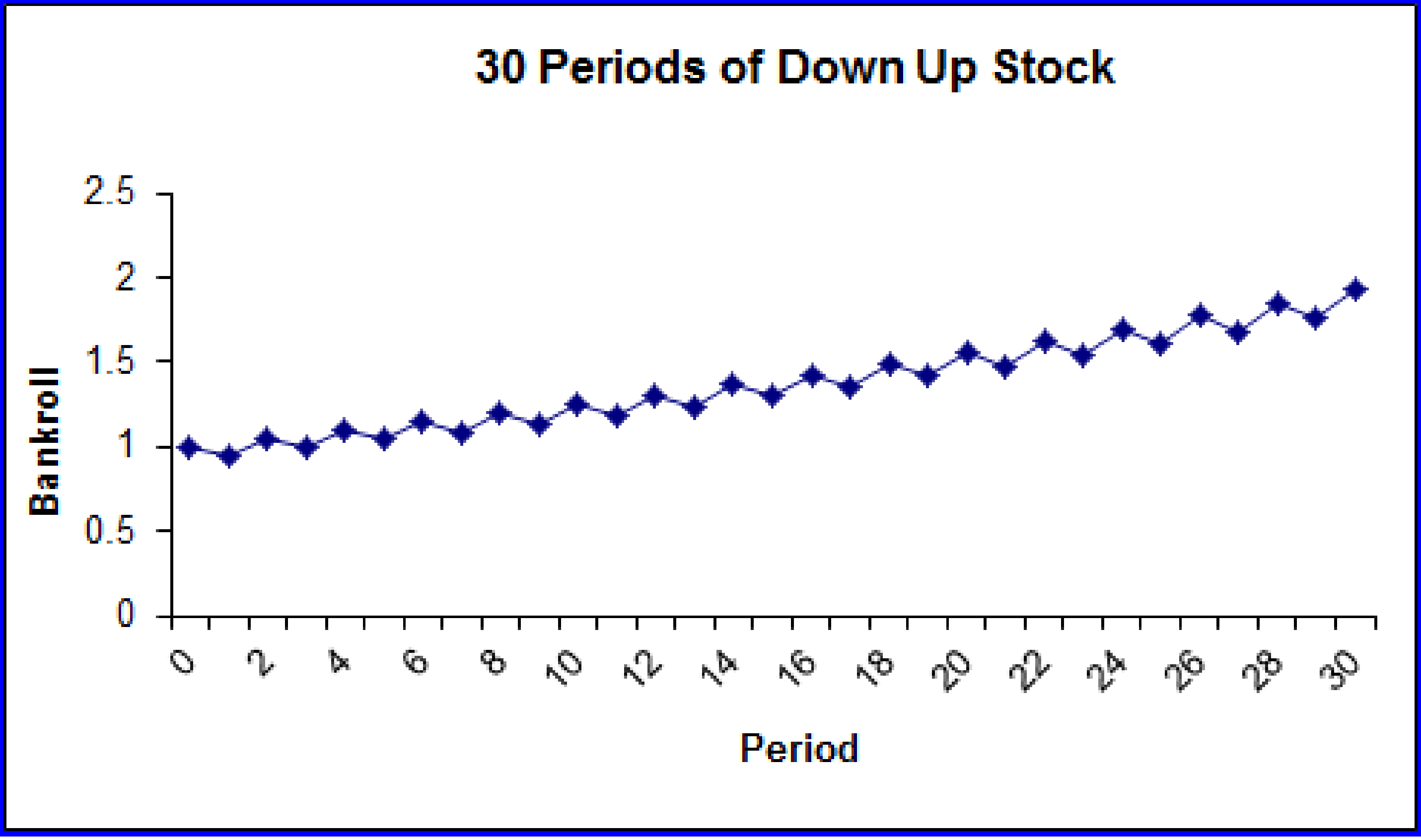
Down Up Stock

- Pretty boring – don't make money this way



Something From Nothing

- What if instead of leaving everything in stock, instead invested 10% of current bankroll and held out other 90%?

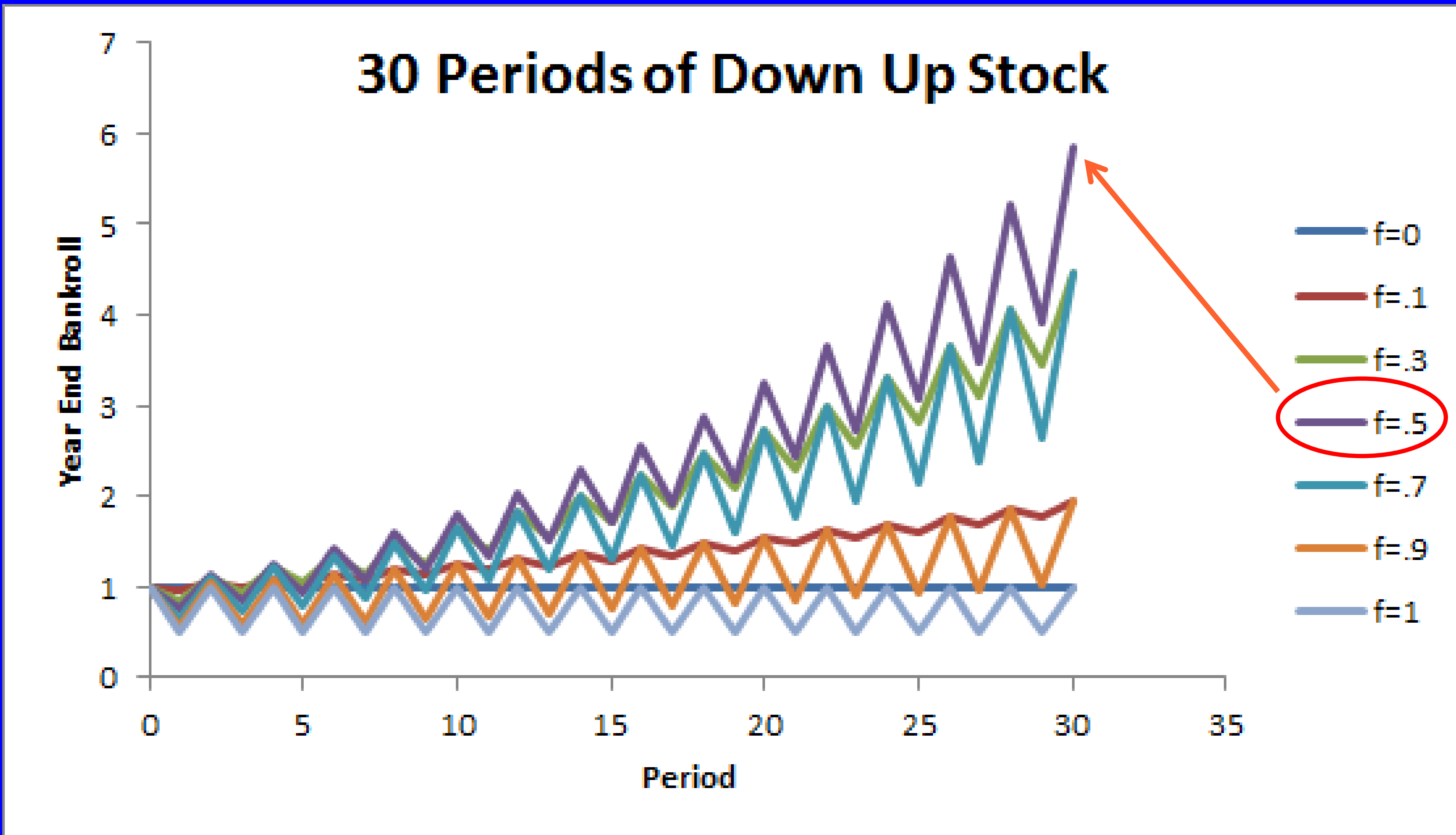


What Just Happened?

- Let B_t = bankroll at end of period t
- In period $t + 1$, invest 10% in down up stock, hold other 90%
- So, at the end of period $t + 1$, you have

  $B_{t+1} = 0.9\, B_t + 0.1\, B_t \times 2^{+1 \text{ or } -1}$
- This is interesting – what if we tried different investment fractions?

Up Down Stock With Different Investment Fractions



Lots of Ways To Get Something From Nothing!

- Among various "fixed fraction" investment strategies, setting the investment fraction $f = 0.5$ gives the best results
- Note that over 30 periods, the stock went down 15 times, and also went up 15 times
- Whenever it went down, lost \$0.50 *per dollar invested*
- Whenever it went up, gained \$1 *per dollar invested*

Aside…

- Of course, if Down Up stock really halved in all odd years and doubled in all even years, then you could just hold out in the odd years, go all in for the even years
- Over 30 years, you'd grow your starting bankroll by a factor of $2^{15} = 32768$!!
- So let's make things a touch more realistic…

Now Suppose Down Up Stock…

- Loses half its value in any period with probability ½
- Doubles in value any period with probability ½
- Results in all time periods independent
- No longer possible to invest only in periods that will double the stock price
- What happens?
- Simulate!!

What's Going On?

- Suppose invest fixed fraction f of bankroll in each period
- Suppose stock goes up u times and down d times
- If stock goes up, then $B_t = B_{t-1}(1+f)$
- If stock goes down, then $B_t = B_{t-1}(1-f/2)$
  - Note – if $f=1$, the bankroll doubles when stock goes up; bankroll halves when stock goes down

What's Going On?

- So, if stock goes up u times and down d times, then after $u + d = n$ periods,

$$B_n = (1 + f)^u (1 - f/2)^d$$

- What is exponential growth rate of bankroll? Call this g
- Need to solve:

$$B_n = (1 + f)^u (1 - f/2)^d = e^{gn}$$

- Solution: $g = \frac{u}{n} \log(1 + f) + \frac{d}{n} \log(1 - f/2)$

What Is The Kelly Criterion?

- Start with exponential growth rate

$$g = \frac{u}{n}\log(1+f) + \frac{d}{n}\log(1-f/2)$$

- Now suppose number of periods *n* gets large
- The *law of large numbers* says that the fraction of periods the stock doubles, *u/n*, goes to the *probability* that the stock doubles (1/2 in this example)
- Similarly, fraction of periods stock halves goes to the probability that stock halves (also 1/2)

Kelly Criterion

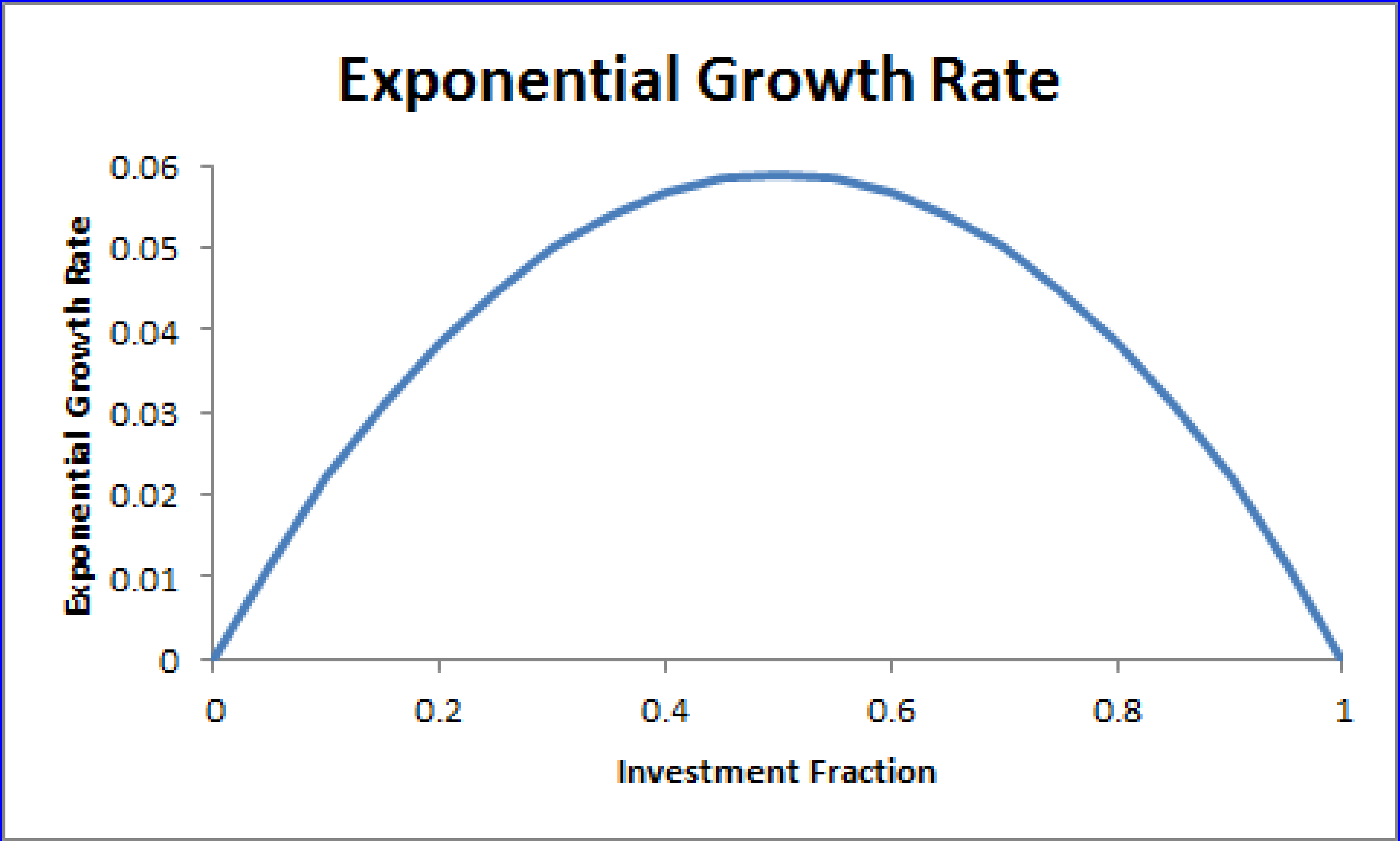
- Pick the investment fraction *f* to *maximize* the expected exponential growth rate
- For random Down Up stock, pick *f* to maximize

$$g = \frac{1}{2}\log(1+f) + \frac{1}{2}\log(1-f/2)$$

- Solve using calculus
- Solution: $f = 1/2$; $g = 0.0589$

Growth Rates for All Fixed Fraction Strategies

- Clear picking $f = 1/2$ gives highest growth rate



What Does This Have To Do With Sports Betting???

- Consider any sports bet (money line, spread, totals, etc.)
- If win bet, payout is $*w per dollar wagered*
- If lose, loss is $*ℓ per dollar wagered*
- You believe chance of winning bet is *p*
- Growth rate given by

$$g = p\log(1+fw) + (1-p)\log(1-f\ell)$$

- Kelly investment fraction given by

$$f = \frac{pw-(1-p)\ell}{w\ell}$$

Kelly Investment Fraction

$$f = \frac{pw-(1-p)\ell}{w\ell}$$

- Suppose you have a money line bet on tonight's Saints at Falcons with Falcons –190
- If win, get \$100 having bet \$190 so w=100/190 = 0.526
- If lose, lose \$190 having bet \$190 so $\ell = 1$
- Suppose you believe Pr{Falcons} = 0.7
- Kelly says, bet 0.1297 or about 13% of your current bankroll

Point Spread Example

$$f = \frac{pw-(1-p)\ell}{w\ell}$$

- Point spread for tonight's game is Falcons by 3.5 points
- Need to bet \$11 to win \$10, so w = 10/11= 0.9091; if lose, lose \$11 on \$11 so ℓ = 1
- Our ratings model estimates that the probability Falcons cover spread = 0.5747
- So, Kelly says bet f = 0.10687

Properties of Kelly Wagering

- Using Kelly, *can't go completely broke!!*
- In long run (i.e. as # bets $n \rightarrow$ infinity), actual return from Kelly strategy beats actual return from any other strategy!
- Expected time to reach a fixed investment goal (e.g. accumulate at least $\$k$) is shorter for Kelly strategy than any other
- Suppose face sequence of favorable binary bets; overall expected growth rate is maximized by investing Kelly fraction in each bet evaluated separately

Has Anyone Really Used This?

The Kelly criterion and its variants: theory and practice in sports, lottery, futures & options trading

The symmetric downside Sharpe ratio and the evaluation of great investors & speculators and their use of the Kelly criterion

William T Ziemba

Alumni Professor at Financial Modeling and Stochastic Optimization, Emeritus,
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Mathematical Finance Seminar
University of Chicago
April 6, 2007

Has Anyone Really Used This?

- According to Bill Ziemba:
  - This talk is a review of the good and bad properties of the Kelly and fractional Kelly strategies and a discussion of their use in practice by great investors and speculators most of whom have become centi-millionaires or billionaires by isolating profitable anomalies and betting on them well with these strategies.
  - The latter include Bill Bentor the Hong Kong racing guru, Ed Thorp, the inventor of blackjack card counting who compiled one of the finest hedge fund records.
  - Both of these gamblers had very smooth, low variance wealth paths.
  - Additionally legendary investors such as John Maynard Keynes (0.8 Kelly) running the King's College Cambridge endowment, George Soros (? Kelly) running the Quantum funds and Warren Buffett (full Kelly) running Berkshire Hathaway had similarly good results but had much more variable wealth paths.

stevanovichcenter.uchicago.edu/seminars/Chicago_**Kelly**.ppt

Consider Ed Thorp

- Excerpts from Thorp, EO. The Kelly Criterion in Blackjack, Sports Betting, and the Stock Market. *Handbook of Asset and Liability Management*, *Volume I*. (SA Zenios and W Ziemba, editors; Elsevier: 2006)

6. Sports betting

In 1993 an outstanding young computer science Ph.D. told me about a successful sports betting system that he had developed. Upon review I was convinced. I made suggestions for minor simplifications and improvements. Then we agreed on a field test. We found a person who was extremely likely to always be regarded by the other sports bettors as a novice. I put up a test bankroll of $50,000 and we used the Kelly system to estimate our bet size.

More Ed Thorp

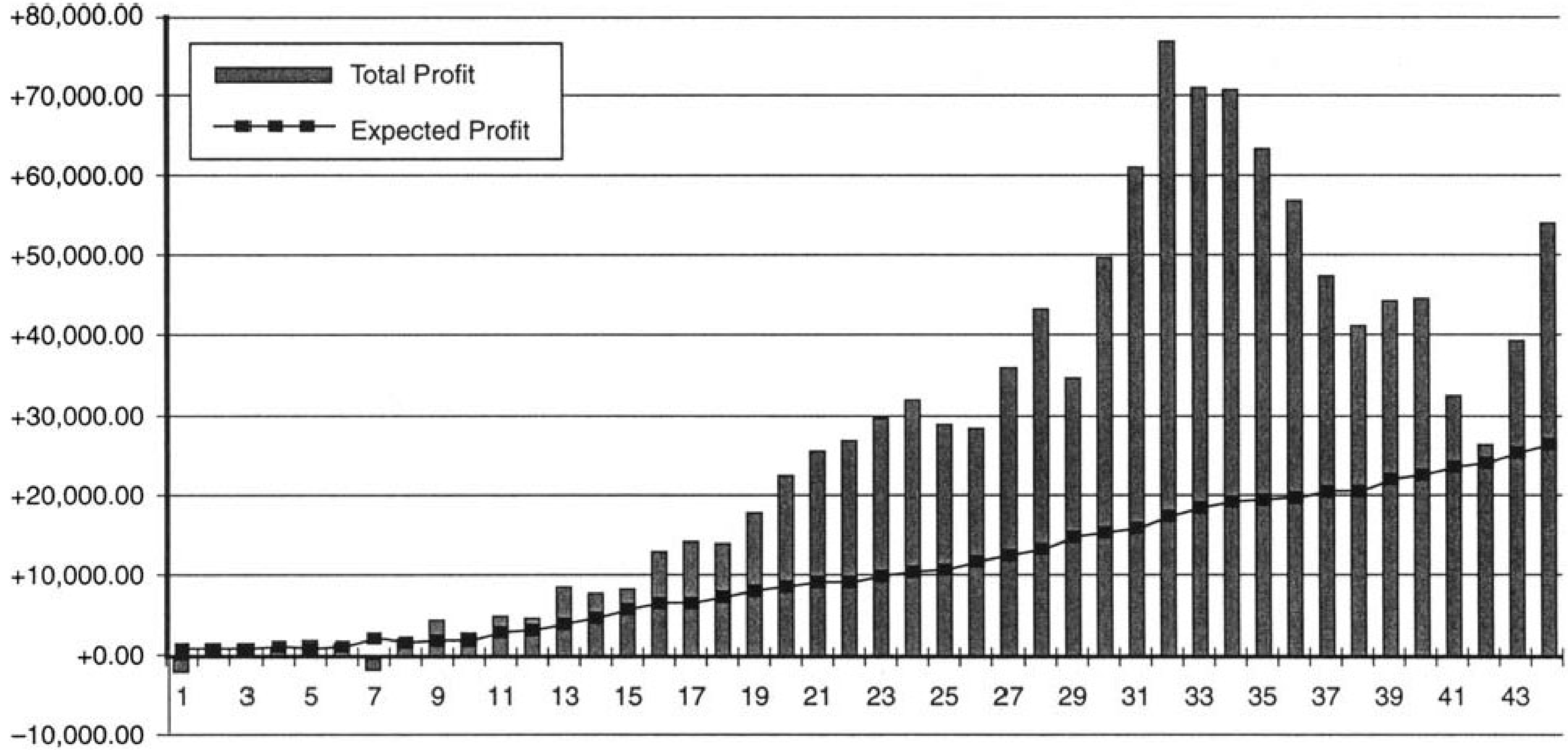

Fig. 3. Betting log Type 2 sports.

Still More Ed Thorp

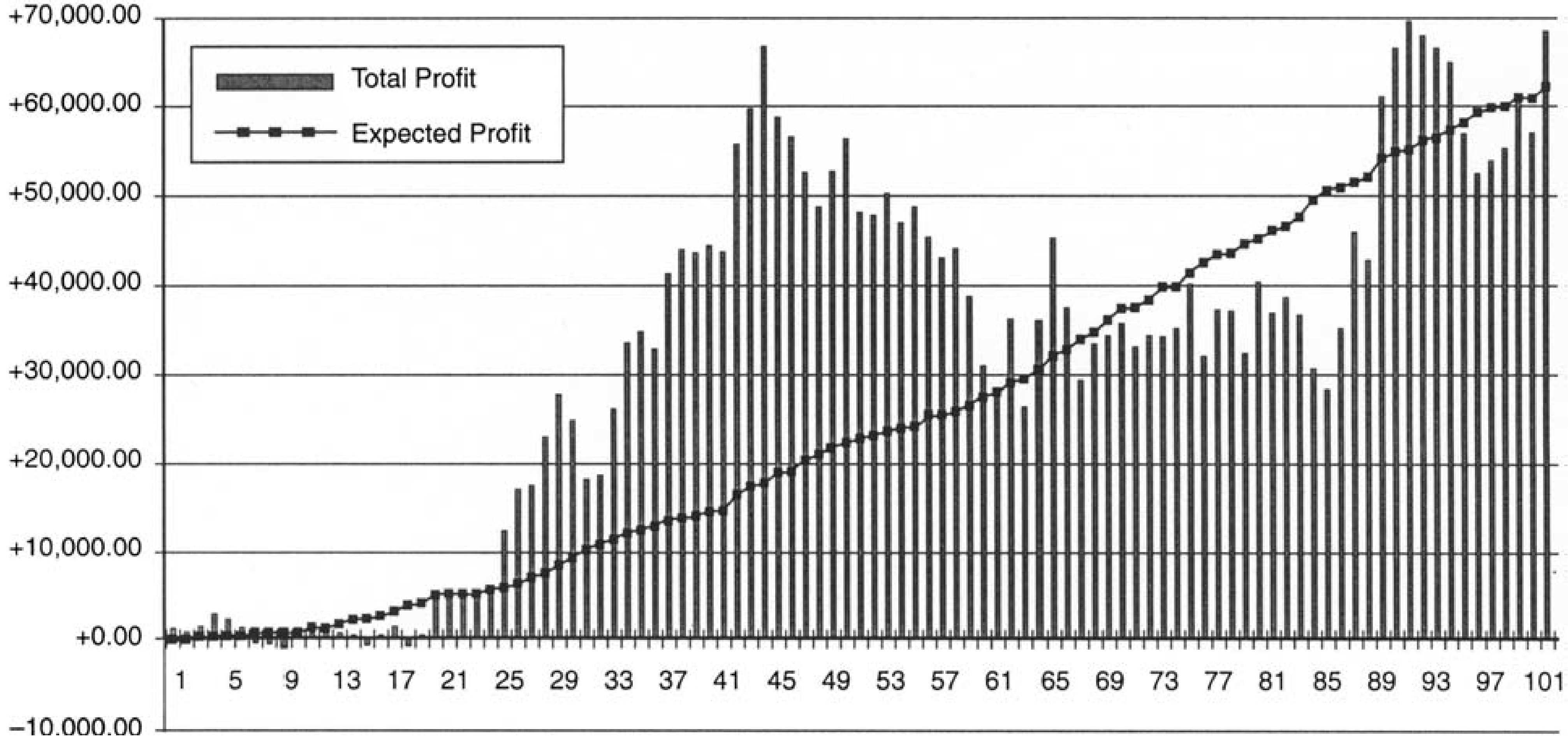

Fig. 4. Betting log Type 1 sports.

Final Words from Ed Thorp

We stopped sports betting after our successful test for reasons including:

(1) It required a person on site in Nevada.

(2) Large amounts of cash and winning tickets had to be transported between casinos. We believed this was very risky. To the sorrow of others, subsequent events confirmed this.

(3) It was not economically competitive with our other operations.

If it becomes possible to place bets telephonically from out of state and to transfer the corresponding funds electronically, we may be back.

Kelly Wagers With More Than Two Outcomes

- Suppose that instead of only two possibilities (win/lose, cover spread/not, etc.), there are multiple mutually exclusive outcomes (e.g. win/place/show in horse race)
- A single bet could pay off different amounts under different outcomes
- Can generalize Kelly for this

m Mutually Exclusive Outcomes

- A wager admits *m* possible outcomes
- If outcome *i* occurs, wager returns x_i dollars per dollar bet (which can be positive or negative between 0 and –1; e.g. –1/2 loses half your investment, while –1 loses all)
- Invest fixed fraction *f* of your bankroll
- Let $z_i = 1$ if outcome *i* occurs; 0 if not
- Post-bet bankroll starting from pre-bet B_0 is

$$B_0 \prod_{i=1}^{m} (1 + f x_i)^{z_i}$$

To Get Kelly Fraction…

- Imagine repeating many bets of this type (life with the horses…)
- Let n_i be the number of times outcome i occurs, and let n be the total number of bets
- Using same trick as in binary case to find exponential growth rate, and noting that law of large numbers says that $n_i / n = p_i$ = probability of outcome i, the expected growth rate is given by

$$g = \sum_{i=1}^{m} p_i \log(1 + f x_i)$$

Kelly Investment Fraction

- So, need to pick f to maximize

$$g = \sum_{i=1}^{m} p_i \log(1 + f x_i)$$

- Sounds like a job for the Solver!
- Let's take a look

A Cute Approximation

- Suppose that fx_i is small enough in absolute value that

$$\frac{1}{1+fx} \approx 1 - fx$$

- e.g. $f = 0.1$, $x = -0.5$
- $1/(1+fx) = 1/(1 - 0.05) = 1.0526$
- $1 - fx = 1 - (-0.05) = 1.05$ – close enough
- Then can prove that
- $$f = \frac{E(X)}{E(X^2)}$$ where $E(X)$, $E(X^2)$ are mean, mean squared payout/$ bet

Also Models A Stock!

- Can view stock return as per dollar invested; infinite number of possible returns but well defined mean and variance (and hence second moment)
- So, either using Solver or cute approximation, suggests how much to invest in a stock!

Simultaneous Kelly Wagers

- All examples thus far have presumed sequential betting
- What if want to bet on three football games that all start at 1 PM on Sunday?
- Need to consider how much to invest in bet on each game at same time
- Need to take all possible combinations of outcomes into account (WWW, WWL, WLW, … , LLL)

Simultaneous Kelly Wagers

- Focus on three game example
- Suppose invest f_1, f_2, and f_3, in bets on games 1, 2, and 3
- Suppose bets pay off x_1, x_2, and x_3, in dollars per dollar wagered when win bet, and lose $1/dollar wagered when lose (so you back the favorite in three money line bets)

Simultaneous Kelly Wagers

- Suppose win all three bets
- Return multiple will be

$$1 + f_1 x_1 + f_2 x_2 + f_3 x_3$$

- Suppose lose bet 1, win bet 2, lose bet 3
- Return multiple will be

$$1 - f_1 + f_2 x_2 - f_3$$

- Assuming the games are independent, letting p_i be chance win bet i, joint event probabilities are just products of marginals

Simultaneous Kelly Wagers

- So Pr{Win all 3 bets} $= p_1 p_2 p_3$
- Pr{Lose 1, Win 2, Lose 3}

  $$= (1 - p_1) p_2 (1 - p_3)$$
- Can now write down log of expected growth rate as sumproduct of joint outcome probabilities and log of return multiples
- Want to maximize as a function of the investment fractions in each bet
- Sounds like another job for Solver!