MGT 721 - Modeling Operational Processes
Spring 2025

Renewal Theory Problem Set: Solutions

These questions are meant to review and extend the concepts introduced in class. You may use any computational assistant you like, including for example Wolfram Alpha's tools for symbolic integration or summation, or even chatGPT, but you are responsible for any errors that might result (especially with chatGPT). You can complete all of these questions without such computational assistants, but nonetheless the option is there.

1. Finding $E(N(t)^2)$, the expected squared number of renewals

(a) Following the same approach taken to evaluate the expected number of renewals (i.e. the renewal function), argue that

$$\begin{aligned} E(N(t)^2) &= \int_0^t E(N(t)^2|T = x) f_T(x) dx \\ &= \int_0^t E[(1 + N(t - x))^2] f_T(x) dx \end{aligned}$$

where T is the basic interarrival time in the renewal process with probability density $f_T(x)$.

With probability $f_T(x)dx$ the first renewal occurs between times $(x, x + dx)$, and given this the remaining number of renewals from x to t is the same as the number of renewals in an interval of duration $t-x$, which is $N(t-x)$. This means that, given a first renewal between x and $x+dx$, the expected squared number of renewals in $(0,t)$ is given by $E[(1+N(t-x))^2]$. Unconditioning over all possible times for a first renewal within $(0,t)$ yields

$$E(N(t)^2) = \int_0^t E[(1+N(t-x))^2] f_T(x)dx$$

(b) Now expand the squared term in the integrand to create a renewal equation of the form

$$E(N(t)^2) = g(t) + \int_0^t E(N(t-x)^2) f_T(x)dx.$$

What is the function $g(t)$?

Note that

$$(1+N(t-x))^2 = 1 + 2N(t-x) + (N(t-x))^2$$

which implies that

$$\begin{aligned}
\int_0^t E[(1+N(t-x))^2] f_T(x)dx &= \int_0^t E[1+2N(t-x)+(N(t-x))^2] f_T(x)dx \\
&= \int_0^t E[2(1+N(t-x)) - 1 + (N(t-x))^2] f_T(x)dx \\
&= 2E(N(t)) - F_T(t) + \int_0^t E[N(t-x)^2] f_T(x)dx.
\end{aligned}$$

This identifies

$$\begin{aligned}
g(t) &= 2E(N(t)) - F_T(t) \\
&= 2\sum_{n=1}^{\infty} F_{T_n}(t) - F_{T_1}(t).
\end{aligned}$$

(c) Solve for $E(N(t)^2)$ by applying the general solution for renewal equations, which in this case is given by

$$E(N(t)^2) = g(t) + \int_0^t g(t-x)h(x)dx$$

where $h(x)$ is the renewal density derived in class.Your final formula should be in terms of the distribution functions $F_{T_n}(t)$ where T_n is the time (from zero) until the n^{th} renewal (and $F_{T_n}(t) = \Pr\{T_n \leq t\}$).

Let's evaluate the integral after remembering that $h(t) = \sum_{n=1}^{\infty} f_{T_n}(t)$:

$$\begin{aligned}
\int_0^t g(t-x)h(x)dx &= \int_0^t (2\sum_{n=1}^{\infty} F_{T_n}(t-x) - F_{T_1}(t-x))(\sum_{j=1}^{\infty} f_{T_j}(x))dx \\
&= 2\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} F_{T_{n+j}}(t) - \sum_{j=1}^{\infty} F_{T_{1+j}}(t).
\end{aligned}$$

Adding back $g(t)$ we get

$$\begin{aligned}
E(N(t)^2) &= 2\sum_{n=1}^{\infty} F_{T_n}(t) + 2\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} F_{T_{n+j}}(t) - \sum_{j=1}^{\infty} F_{T_j}(t) \\
&= \sum_{n=1}^{\infty}(2n-1)F_{T_n}(t).
\end{aligned}$$

(d) As a check, solve for $E(N(t)^2)$ using the standard formula from basic probability, namely

$$E(N(t)^2) = \sum_{n=1}^{\infty} n^2 \Pr\{N(t) = n\}$$

where from class recall that

$$\Pr\{N(t) = n\} = \Pr\{N(t) \geq n\} - \Pr\{N(t) \geq n+1\}$$

and also that

$$\Pr\{N(t) \geq n\} = \Pr\{T_n \leq t\}$$

Of course, once you have a formula for $E(N(t)^2)$, you can compute $Var(N(t)) = E(N(t)^2) - [E(N(t)]^2$.

Plug and play! We have

$$\begin{aligned}
E(N(t)^2) &= \sum_{n=1}^{\infty} n^2 \Pr\{N(t) = n\} \\
&= \sum_{n=1}^{\infty} n^2 [\Pr\{N(t) \geq n\} - \Pr\{N(t) \geq n+1\}] \\
&= \sum_{n=1}^{\infty} n^2 [F_{T_n}(t) - F_{T_{n+1}}(t)] \\
&= F_{T_1}(t) + (4-1)F_{T_2}(t) + (9-4)F_{T_3}(t) + ... \\
&= \sum_{n=1}^{\infty} (2n-1) F_{T_n}(t).
\end{aligned}$$

Same answer!

2. Uniform Renewal Process

Suppose that one has a renewal process with interarrival times uniformly distributed between 0 and 1. In class we found that for $t < 1$, the expected number of renewals is given by $E(N(t)) = e^t - 1$. Using what you just learned above, for $t < 1$ find $E(N(t)^2)$ for this renewal process. Whether you use the approach of part (c) or part (d), show that your result satisfies the equation in part (b).

Let's start with the approach of part (c), which will simplify in this case since we already know that $f_T(t) = 1$, $F_T(t) = t$, $E(N(t)) = e^t - 1$, and thus $h(t) = e^t$ for $0 \leq t \leq 1$. First we have that

$$\begin{aligned}
g(t) &= 2E(N(t)) - F_T(t) \\
&= 2(e^t - 1) - t \text{ for } 0 \leq t \leq 1
\end{aligned}$$

Plugging this into the renewal equation gives

$$\begin{aligned}
E(N(t)^2) &= 2(e^t - 1) - t + \int_0^t [2(e^{t-x} - 1) - (t-x)] e^x dx \\
&= 2te^t - e^t + 1.
\end{aligned}$$

Just for kicks let's try the approach of part (d). Note that for $0 \leq t \leq 1$,

$$F_{T_n}(t) = \frac{t^n}{n!}$$

(this is easy to prove), so pluging into our infinite sum we get

$$\begin{aligned} E(N(t)^2) &= \sum_{n=1}^{\infty}(2n-1)\frac{t^n}{n!} \\ &= 2te^t - e^t + 1 \end{aligned}$$

as before.

Now the defining renewal equation in part (b) is

$$E(N(t)^2) = g(t) + \int_0^t E(N(t-x)^2) f_T(x)dx.$$

So, plugging in our newly-discovered results for $E(N(t)^2)$ we get

$$\begin{aligned} E(N(t)^2) &= 2(e^t - 1) - t + \int_0^t (2(t-x)e^{t-x} - e^{t-x} + 1)dx \\ &= 2te^t - e^t + 1. \end{aligned}$$

Hey! It works!

3. The Die Is Cast!

Imagine taking a single die that, after rolling, shows (via dots) the integers 1 through 6 inclusive, each with equal probability 1/6. Now let S_n equal the sum of the numbers obtained from rolling the die n times. So if you roll 3 followed by 1 followed by 5, you would have $S_1 = 3$. $S_2 = 3 + 1 = 4$, and $S_3 = 3 + 1 + 5 = 9$. What is the probability that for some integer n, $S_n = 2025$? That is, what is the probability that after *some* number of rolls, the running sum of of the numbers from each roll will *exactly* equal 2025? Now, there are exact and nearly exact ways to answer this question, but your job is to produce a ridiculously simple approximation based on – what else? Renewal theory! Clearly explain your answer. If you want to produce either the exact (or very nearly exact) answer patiently using basic methods, you can, but if you choose to do so, do it only to verify your approximation from renewal theory.

To see why this is such an easy problem, imagine the number shown on the die as an interarrival time. The sum of the numbers on the die after n rolls, S_n, then corresponds exactly to what we have been calling T_n, which is the time of the n^{th} renewal. The question is thus just asking for the probability that a renewal occurs at time 2025. You get an actual probability here since the interarrival times are all discrete – 1, 2, ..., 6 each with probability 1/6. Now the expected number of renewals that occur within time t as t gets large is just t/τ where τ is the expected interarrival time. In this problem, $\tau = 3.5$. Why? Well, what is $(1+2+3+4+5+6)/6$? 3.5 (!). This means that the expected number of renewals by time t is growing as $t/3.5$, while the renewal density $h(t) = 1/3.5$ (and recall from class that in general the renewal density for large t in general just equals $1/\tau$). Now recall the interpretation of the renewal density – in continuous time, $h(t)dt$ is the probability of a renewal between t and $t + dt$. But in discrete time, $h(t)$ is the *probability* that a renewal occurs *at* time t! So, to a very close approximation, the probability that after some number of rolls, the running sum of the numbers from each roll will exactly equal 2025 is just equal to $1/3.5 = 0.285\,7$.

Now for a more detailed excellent approximation. Letting n indicate the number of rolls of the dice to reach 2025, we immediately see that $n \leq 2025$ (that would occur from getting a 1 on every roll), and also that $n \geq 338$ (if you got a 6 on 337 consecutive rolls you would have a total of $6 \times 337 = 2022$, so you'd need a 338^{th} roll (and a 3 at that) to get to 2025.

Let $p(n) \equiv \Pr\{S_n = 2025\}$, that is the probability of hitting 2025 exactly after n rolls for n between 338 and 2025. Then the probability of hitting 2025 exactly after *some* number of rolls is just

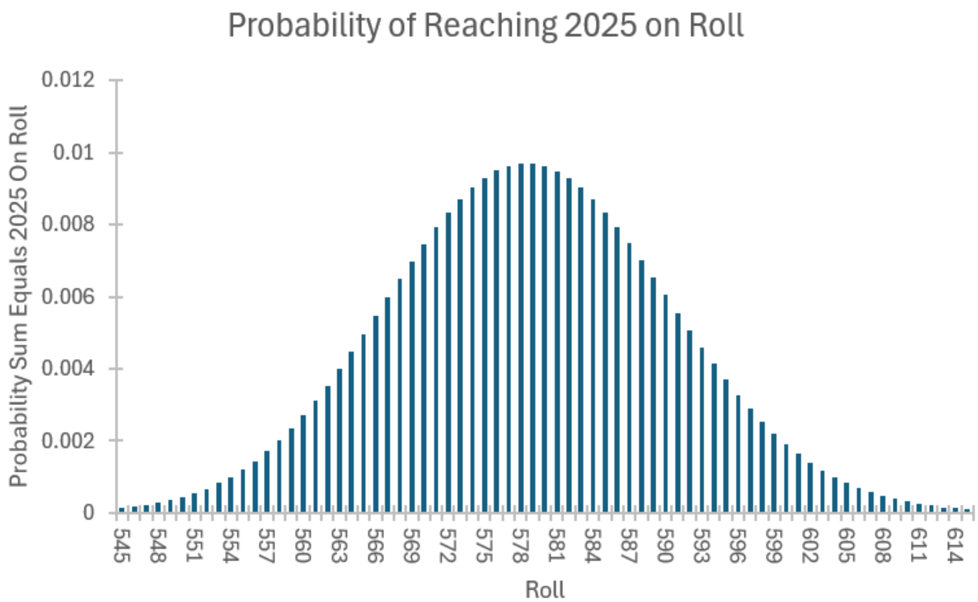
$$\Pr\{Hit\ 2025\ Exactly\} = \sum_{n=338}^{2025} p(n).$$

From the central limit theorem, we know that S_n is approximately normal with mean $3.5n$ and variance $(35/12)n$ (where 35/12 is the variance of a discrete random variable taking on the values 1 though 6 each with probability 1/6). With a slight abuse of notation, we'll also let S_n denote the approximating normal random variable, and from that we can write

$$\begin{aligned} p(n) &= \Pr\{2024.5 \leq S_n \leq 2025.5\} \\ &= \Pr\{Z \leq \frac{2025.5 - 3.5n}{\sqrt{\frac{35}{12}n}}\} - \Pr\{Z \leq \frac{2024.5 - 3.5n}{\sqrt{\frac{35}{12}n}}\} \end{aligned}$$

where Z is the standard normal random variable. Do this for all n between 338 and 2025 and add them up!

Below is a plot of $p(n)$ as computed using the normal approximation.



Adding up the heights of all these bars, that is computing $\sum_{n=338}^{2025} p(n)$ yields ... 0.2857 (!!) Now what was easier – using renewal theory to get 1/3.5, or using the normal approximation?

If you really want to torture yourself, you could compute this exactly by getting the exact probability distribution of S_n. This would be done via recursion. The basic recursion is

$$\Pr\{S_n = s\} = \sum_{k=1}^{6} \Pr\{S_{n-1} = s - k\} \times \frac{1}{6}.$$

You of course initialize with the uniform distribution for S_1 taking on possible values of 1 through 6, and then you iterate over s to get the entire exact distribution of S_n up to $n = 2025$. Now define $p(n) = \Pr\{S_n = 2025\}$ using the exact distribution, and you get your final answer from $\sum_{n=338}^{2025} p(n)$. Go ahead and program this if you like, but guess what you'll discover? The answer is 0.2857!

4. The A B Shuttle (not as much fun as Springsteen's E Street Shuffle but...)

A single shuttle goes back and forth between locations A and B. The time required to travel from A to B is a random variable T. As it turns out, the time required to travel back from B to A is identically distributed with the same distribution as the time required to travel from from A to B. Let the common random variable for the travel time from A to B (or from B to A) have probability distribution function

$$F_T(t) = \Pr\{T \le t\} = \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{\lambda t}} e^{-u^2} du, \ \ \lambda > 0$$

Some of you might recognize $F_T(t)$ via the *error function*

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-u^2} du$$

which makes

$$F_T(t) = \text{erf}(\sqrt{\lambda t}).$$

But I digress.

(a) What is the probability density function $f_T(t)$ associated with the time required for a one way trip from A to B (or from B to A)?

We need to remember our basic probability here:

$$\begin{aligned} f_T(t) &= \frac{d}{dt} F_T(t) \\ &= \frac{d}{dt}\frac{2}{\sqrt{\pi}} \int_0^{\sqrt{\lambda t}} e^{-u^2} du \\ &= \frac{2}{\sqrt{\pi}}(\frac{d}{dt}\sqrt{\lambda t})e^{-(\sqrt{\lambda t})^2} \\ &= \sqrt{\frac{\lambda}{\pi t}} e^{-\lambda t}, \ t > 0. \end{aligned}$$

(b) What is the expected time for the shuttle to travel one way from A to B (or vice versa), that is what is $E(T)$?

This follows from the density function as

$$E(T) = \int_0^\infty t \times \sqrt{\frac{\lambda}{\pi t}} e^{-\lambda t} dt = \frac{1}{2\lambda}$$

(c) What is the variance of the time for the shuttle to make a one way trip, that is, what is $Var(T)$?

Let's use our usual formula: $Var(T) = E(T^2) - [E(T)]^2$ and note that

$$E(T^2) = \int_0^\infty t^2 \times \sqrt{\frac{\lambda}{\pi t}} e^{-\lambda t} dt = \frac{3}{4\lambda^2}$$

from which we obtain

$$Var(T) = \frac{3}{4\lambda^2} - \left(\frac{1}{2\lambda}\right)^2 = \frac{1}{2\lambda^2}$$

(d) You want to catch the shuttle at A to travel to B. Relative to the timing of the shuttle, you arrive at A at a random time. Assume that the time the shuttle spends at A or B is negligible relative to the travel time from A to B or B to A.

(i) What is the probability that at the time you arrive at A, the shuttle is traveling between A and B? What is the probability the shuttle is traveling between B and A?

The probability distribution of the time from A to B is the same as the distribution from B to A, so arriving at A at a random time gives a 50% chance the shuttle is going from A and B, and a 50% chance the shuttle is going from B to A.

(ii) Conditional on the shuttle traveling between A and B at the time of your arrival at A, what is your expected waiting time until the shuttle arrives at A?

Since you arrived at random relative to the shuttle process, if you are told that the shuttle is traveling between A and B, your expected waiting time until the shuttle arrives at A is the sum of two things: the expected remaining time in the trip from A to B, and the unconditional expected time from B to A. We know already from part (b) that the unconditional expected time from B to A just equals $1/(2\lambda)$. The expected remaining time from your arrival at A until the shuttle arrives at B is just the expected forward recurrence time given random incidence on random variable T. From class we know that the expected forward recurrence time is equal to

$$E(T^*) = \frac{E(T^2)}{2E(T)} = \frac{3/\left(4\lambda^2\right)}{2 \times 1/(2\lambda)} = \frac{3}{4\lambda}$$

The expected total wait from your arrival at A until the shuttle arrives at A, conditional on the shuttle traveling between A and B at the time of your arrival, is then given by

$$E(W|\text{shuttle going from A to B when you arrive at A}) = \frac{3}{4\lambda} + \frac{1}{2\lambda} = \frac{5}{4\lambda}$$

(iii) From the time of your arrival at A, what is your (unconditional) expected waiting time until the shuttle arrives?

With probability 1/2, you arrive while the shuttle is moving from A to B which gives a conditional expected wait of $5/(4\lambda)$. But also with probability 1/2, you arrive while the shuttle is already en route from B back to A, and in this case your expected wait is only $3/(4\lambda)$.Therefore your overall expected wait given that you randomly arrive at A is equal to

$$E(W) = \frac{1}{2} \times \frac{5}{4\lambda} + \frac{1}{2} \times \frac{3}{4\lambda} = \frac{1}{\lambda}$$

(e) Now, again assuming that the time the shuttle spends at A or B is negligible, the time required for the shuttle to make a round trip is just the sum of two one way trips, that is, if R is the duration of a round trip, then

$$R = T_{AB} + T_{BA}$$

where T_{AB} and T_{BA} are independently and identically distributed as random variable T above. What is the probability density $f_R(t)$ for the time required for a round trip? What is $E(R)$? What is $Var(R)$?

Working directly with the fact that T_{AB} and T_{BA} are independently and identically distributed as random variable T we obtain

$$\begin{aligned} f_R(t) &= \int_0^t f_T(s) \times f_T(t-s)ds \\ &= \int_0^t \sqrt{\frac{\lambda}{\pi s}} e^{-\lambda s} \times \sqrt{\frac{\lambda}{\pi(t-s)}} e^{-\lambda(t-s)} ds \\ &= \frac{\lambda}{\pi} e^{-\lambda t} \int_0^t \frac{1}{\sqrt{s \times (t-s)}} ds \\ &= \frac{\lambda}{\pi} e^{-\lambda t} \times \pi \\ &= \lambda e^{-\lambda t}, \quad t > 0 \end{aligned}$$

How cool is that? We just learned that R has an exponential distribution! Usually we think of adding up exponentials to get stuff, but before this problem, have you ever seen the exponential emerge as the *sum* of random variables?

You might have anticipated this if you recognized $f_T(t)$ as a special case of the gamma density. A random variable X has a gamma distribution with positive parameters α and λ if

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0.$$

Two facts about gamma random variables with this density: (i) $E(X) = \alpha/\lambda$ and $Var(X) = \alpha/\lambda^2$, and (ii) if X_j is gamma distributed (α_j, λ) for $j = 1, 2, ..., n$ and all of the $X_j's$ are mutually independent, then $S_n = \sum_{j=1}^n X_j$ is also gamma distributed with parameters $(\sum_{j=1}^n \alpha_j, \lambda)$. In the present problem, T has a gamma distribution with $\alpha = 1/2$, while R has a gamma distribution with $\alpha = 1/2 + 1/2 = 1$. Nice how this all hangs together, no?

Anyway, now that we know R is exponentially distributed with rate λ, it follows immediately that $E(R) = 1/\lambda$ while $Var(R) = 1/\lambda^2$.

(f) What is the probability density function for the time you will have to wait from your arrival at A until the shuttle next arrives at A? Verify using this density that your expected waiting time for the shuttle after you randomly arrive at A indeed equals your result from (d) (iii) above.

OK – now if you arrive at random relative to the shuttle, all you need to know is that the interarrival times for the shuttle at A are exponentially distributed with rate λ. The exponential distribution is of course *memoryless*, so the remaining time until the shuttle arrives after your random arrival at A also equals $1/\lambda$. Surprise – part (d) (iii) above gave exactly the same result, though it was more work to get it, no?

(g) Let $N(t)$ denote the total number of completed shuttle round trips as of time t when the shuttle begins operating from A at time zero. What is the probability distribution of $N(t)$?

This is a gift! Knowing as you now do that the interarrival time distribution at A is exponential with rate λ, the number of completed shuttle round trips as of time t just follows a Poisson distribution with mean λt, that is,

$$\Pr\{N(t) = n\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \;\; n = 0, 1, 2, ...$$

5. Treatment and Monitoring

A seriously ill patient has just arrived at the hospital at time 0. This patient immediately enters treatment. Any treatment episode requires U time units where U is a non-negative continuous random variable with expected value $E(U)$. All treatment episodes are iid . Following any treatment episode, the patient is discharged from the hospital with probability $1 - \alpha$, while with probability α the patient is held at the hospital for a monitoring period that requires V time units where V is a non-negative continuous random variable with expected value $E(V)$. All monitoring episodes are iid, and all treatment and monitoring episodes are mutually independent. Following any monitoring episode, the patient is discharged from the hospital with probability $1 - \beta$, but with probability β the patient is returned for a subsequent treatment episode.

(a) Let $f_U(u)$ and $f_V(v)$ denote the probability density functions for random variables U (treatment duration) and V (monitoring duration) respectively, and let $\phi(t)$ be the probability that a seriously ill patient who arrived at the hospital at time 0 is in treatment at time t. Produce an equation that if solved will reveal $\phi(t)$. Your equation should be in the spirit of renewal theory (or maybe in the advanced spirit of the repetition method), and must depend upon α, β, $f_U(u)$ and $f_V(v)$. You don't need to solve this equation; just formulate it.

For a patient who arrived at time 0 to be in treatment at time t, either the patient has been in treatment since arrival (which happens with probability $\Pr\{U > t\}$), **or**: at some time $u < t$, the patient completes treatment (with probability $f_U(u)du$), proceeds to monitoring (with probability α; note that if the patient did not proceed to monitoring then the patient would have been discharged and hence could not be in treatment at time t), completes monitoring at time $u+v < t$ (with probability $f_V(v)dv$), returns to treatment (with probability β; note that if the patient did not return to treatment then the patient would have been discharged and hence could not be in treatment at time t), and is then found in treatment $t-u-v$ time units later at time t as required (with probability $\phi(t-u-v)$).

Taken together the arguments above yield

$$\phi(t) = \Pr\{U > t\} + \int_{u=0}^{t} f_U(u)\alpha \int_{v=0}^{t-u} f_V(v)\beta\phi(t-u-v)dvdu, \ t > 0.$$

(b) From the moment of arrival at the hospital (time 0), let τ_T denote the expected total time a seriously ill patient will spend in treatment episodes until discharge. How would you determine τ_T if you knew $\phi(t)$?

Let random variable $X(t) = 1$ if the patient is in treatment at time t after arrival to the hospital, and zero otherwise. Then the expected total time the patient will spend in treatment episodes until discharge is given by

$$\tau_T = E[\int_0^\infty X(t)dt] = \int_0^\infty E[X(t)]dt = \int_0^\infty \phi(t)dt$$

since the probability that the patient is in treatment at time t after arrival to the hospital is equal to $\phi(t) = \Pr\{X(t) = 1\} = E(X(t))$ since $X(t)$ is a Bernoulli random variable.

(c) Let random variable N denote the total number of treatment episodes until discharge required by a newly arriving patient at the hospital. Produce an explicit formula for $\Pr\{N = n\}$ for $n = 1, 2, 3, ...$, that is, determine the probability distribution for random variable N.

The number of treatment episodes N can be determined as follows: suppose that a patient has just completed n treatment episodes, $n = 1, 2, 3, ...$ With probability $\alpha\beta$, the patient will complete another treatment episode (having first gone through monitoring and then back to treatment), while with probability $1-\alpha\beta$ the patient will have been discharged after treatment (with probability $1-\alpha$) or sent to monitoring and discharged afterwards (with probability $\alpha(1-\beta)$; note that $1-\alpha+\alpha(1-\beta)=1-\alpha\beta$). Combining these observations, we see that the probability a patient experiences exactly n treatment episodes equals the probability that the patient repeats treatment $n-1$ times and is discharged after the n^{th} episode. Thus,

$$\Pr\{N=n\}=(\alpha\beta)^{n-1}(1-\alpha\beta),\ n=1,2,3...$$

That is, the random variable N has a geometric distribution with stopping probability $(1-\alpha\beta)$.

(d) Now produce an explicit formula for τ_T, the expected total time spent in treatment, that does not require knowledge of $\phi(t)$.

First, note that in expectation each treatment episode requires $E(U)$ time units. Second, from the geometric distribution in part (c), that the expected number of treatment episodes is given by

$$E(N)=\frac{1}{1-\alpha\beta}.$$

The expected total time in treatment τ_T is then given by

$$\tau_T=\frac{E(U)}{1-\alpha\beta}.$$

More formally, if random variables U_i are i.i.d. with probability density $f_U(u)$ for $i=1,2,3,...$, the number of treatment episodes N has the geometric distribution from part (c), and N is independent of all of the U_i's, then

$$\tau_T=E\left[\sum_{i=1}^{N}U_i\right]=E(N)E(U)=\frac{E(U)}{1-\alpha\beta}$$

as claimed.

As a matter of interest, here is how you could have obtained τ_T directly from the defining equation for $\phi(t)$ in part (a). In part (b) you discovered that $\tau_T=\int_0^\infty\phi(t)dt$. This means that $\phi(t)/\tau_T$ is a probability density function for some random variable we will call W. Now multiply and divide the second part of the equation for $\phi(t)$ by τ_T to obtain the expression

$$\alpha\beta\tau_T \int_{u=0}^{t} f_U(u) \int_{v=0}^{t-u} f_V(v) \frac{\phi(t-u-v)}{\tau_T} dvdu = \alpha\beta\tau_T \int_{u=0}^{t} f_U(u) \int_{v=0}^{t-u} f_V(v) f_W(t-u-v) dvdu$$

where $f_W(w) = \phi(w)/\tau_T$ is the probability density for W. Recognize the integral expression as the convolution of the probability densities for random variables U, V and W, that is,

$$\int_{u=0}^{t} f_U(u) \int_{v=0}^{t-u} f_V(v) f_W(t-u-v) dvdu = f_{U+V+W}(t),\ t > 0$$

which is the probability density function for the sum of U, V and W. The equation for $\phi(t)$ can thus be written as

$$\phi(t) = \Pr\{U > t\} + \alpha\beta\tau_T f_{U+V+W}(t).$$

Integrating both sides of this equation yields

$$\begin{aligned} \tau_T &= \int_0^\infty \phi(t)dt \\ &= \int_0^\infty \Pr\{U > t\}dt + \int_0^\infty \alpha\beta\tau_T f_{U+V+W}(t)dt \\ &= E(U) + \alpha\beta\tau_T \end{aligned}$$

where $E(U)$ follows from "integrating the tail" and $\alpha\beta\tau_T$ follows from noting that the integral of $f_{U+V+W}(t)$ must equal unity since it is a probability density! We thus have the simple identity

$$\tau_T = E(U) + \alpha\beta\tau_T$$

from which we again obtain

$$\tau_T = \frac{E(U)}{1-\alpha\beta}$$

as claimed.